COMPOSITION OPERATORS ON UNIFORM ALGEBRAS, ESSENTIAL NORMS, AND HYPERBOLICALLY BOUNDED SETS

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ABSTRACT. Let A be a uniform algebra, and let ϕ be a self-map of the spectrum M_A of A that induces a composition operator C_ϕ on A. The object of this paper is to relate the notion of "hyperbolic boundedness" introduced in [GGL] to the essential spectrum of C_ϕ. It is shown that the essential spectral radius of C_ϕ is strictly less than 1 if and only if the image of M_A under some iterate ϕ^n of ϕ is hyperbolically bounded. The set of composition operators is partitioned into "hyperbolic vicinities" that are clopen with respect to the essential operator norm. This partition is related to the analogous partition with respect to the uniform operator norm.

Some attention has been paid over the years to homomorphisms of uniform algebras, particularly to compact homomorphisms. A unital homomorphism of a uniform algebra A can be realized as a composition operator C_ϕ given by $f \mapsto f \circ \phi$, where ϕ is a self-map of the spectrum M_A of A. H. Kamowitz [Ka] obtained in 1980 a fundamental theorem on compact composition operators asserting the existence of an attracting fixed point for the underlying map ϕ of the spectrum. In his path-breaking 1996 thesis written at Karlsruhe under R. Mortini, U. Klein exploited systematically the contractive properties of the pseudohyperbolic metric on the spectrum to shed light on the Kamowitz theorem and to obtain substantial generalizations. (For an expository account of Klein's thesis, see [Ga4].) In order to clarify the natural boundaries of the fixed point theorem along the lines laid out by Klein, we introduced and explored in [GGL] a notion of "hyperbolic boundedness". In this paper we continue the work in [GGL] by relating hyperbolic boundedness to the essential spectra of composition operators and to the essential norm of the difference of composition operators.

Our immediate motivation stems from the thesis of L. Zheng [Zh], in which he shows that for an analytic self-map ϕ of $\mathbb{D}$, the composition operator C_ϕ on $H^\infty(\mathbb{D})$ has essential spectral radius either 0 or 1. Under supplementary conditions (see later) the essential spectrum fills out the closed unit disk $\overline{\mathbb{D}}$. We extend Zheng's results to the setting of homomorphisms between uniform algebras. P. Gorkin and R. Mortini [GM] have also extended Zheng's work, and our Lemma 2.1 and Corollary 2.2 below are closely related to their Theorems 8 and 9. For related work, see also [HIZ].

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The paper is organized as follows. In Section 2 we characterize composition operators with essential spectral radius 1. In Section 3 we introduce norm vicinities and hyperbolic vicinities of composition operators, and we give conditions for operators to belong to the same hyperbolic vicinity. In Section 4 we show that two composition operators belong to the same hyperbolic vicinity if and only if their restrictions to some subalgebra of finite codimension belong to the same norm vicinity. In Section 5 we show that hyperbolic vicinities are open with respect to the essential operator norm. In Section 6 we determine the hyperbolic vicinity of the identity operator and we show that the identity operator is essentially isolated in the set of composition operators on an algebra whose spectrum has no isolated points.

1. Background

Let A be a uniform algebra, with spectrum M_A. We regard A as an algebra of continuous functions on M_A, so that A is a closed unital subalgebra of $C(M_A)$. The pseudohyperbolic metric $\rho_A(x,y)$ on the spectrum M_A is defined by

$$(1) \qquad \rho_A(x,y) \;=\; \sup\{\rho(f(x),f(y)) : f \in A, ||f|| < 1\}, \qquad x,y \in M_A,$$

where ρ is the pseudohyperbolic metric on the open unit disk $\mathbb{D}$ in the complex plane. The pseudohyperbolic metric on M_A is expressed in terms of the norm of the evaluation functional at x on the null-space of the evaluation functional at y by

$$(2) \quad \rho_A(x,y) \;=\; ||x|_{y^{-1}(0)}|| \;=\; \sup\{|f(x)| : f \in A, ||f|| < 1, f(y) = 0\}, \qquad x,y \in M_A.$$

Evidently $\rho_A(x,y) \le 1$. The pseudohyperbolic metric on M_A satisfies König's inequality

$$(3) \qquad \rho_A(x,y) \;\le\; \frac{\rho_A(x,u)+\rho_A(u,y)}{1+\rho_A(x,u)\rho_A(u,y)}, \qquad x,u,y \in M_A.$$

¿From König's inequality it is easy to see that any two open pseudohyperbolic balls in M_A of radius 1 either are disjoint or coincide. These open balls of radius 1 are the *Gleason parts* of A. (See Chapter VI of [Ga1].) It is easy to check that

$$\frac{||x-y||}{2} \;\le\; \rho_A(x,y) \;\le\; ||x-y||, \qquad x,y \in M_A,$$

so that convergence in the pseudohyperbolic metric of M_A is tantamount to convergence in the norm of A^*. The precise relationship between the pseudohyperbolic metric and the norm in A^* is given by

$$(4) \quad ||x-y|| \;=\; \frac{2}{\rho_A(x,y)}\left(1-\sqrt{1-\rho_A(x,y)^2}\right), \qquad \rho_A(x,y) \;=\; \frac{4||x-y||}{4+||x-y||^2},$$

where $x,y \in M_A$. (See e.g. [Kö]).

For general background on uniform algebras we refer to [Ga1].

The bidual A^{**} of A is also a uniform algebra. For a description of the bidual of A, see [Ga3]. The evaluation functionals at points of M_A extend uniquely to be weak-star continuous multiplicative functionals on A^{**}, so we can regard M_A as a subset of the spectrum of A^{**}, and we can regard A as a subalgebra of A^{**}. The restrictions of the functions in A^{**} to M_A are the pointwise limits of bounded nets in A.

Following [GGL], we define a subset E of M_A to be *hyperbolically bounded* if it is contained in a finite union of pseudohyperbolic balls whose radii are strictly less than 1. Each such ball is contained in a single Gleason part, so that a hyperbolically bounded subset of M_A meets only a finite number of Gleason parts of M_A.

We will make use of the following result, which is Theorem 2.5 in [GGL].

Theorem 1.1. *Let A be a uniform algebra, with spectrum M_A and bidual A^{**}. Let E be a subset of M_A that is not hyperbolically bounded. Then for each $\varepsilon > 0$, there are a sequence of points $\{x_j\}_{j=1}^{\infty}$ in E and a sequence of functions $\{F_k\}_{k=1}^{\infty}$ in A^{**} such that $F_k(x_j) = 0$ for $j \neq k$, $F_k(x_k) = 1$, and $\sum_{k=1}^{\infty} |F_k| \leq 1 + \varepsilon$ on $M_{A^{**}}$.*

Note that the sequence $\{F_k\}$ in the theorem converges weakly in A^{**} to 0. In fact, for a sequence of functions $\{f_k\}$ in a subspace of $C(X)$, the condition $\sum |f_k(x)| \leq M < \infty$ holds if and only if the correspondence $e_k \mapsto f_k$ extends to a continuous linear operator from the sequence space c_0 into the subspace. (See e.g. [Da].) Since the standard basis $\{e_k\}$ for c_0 tends weakly to 0, so does $\{f_k\}$. (This can also be seen by representing a continuous linear functional on $C(X)$ by a measure and applying the Lebesgue dominated convergence theorem.)

A *composition operator* from a uniform algebra A to a uniform algebra B is an operator of the form $C_\phi : f \mapsto f \circ \phi$, where $\phi : M_B \to M_A$ is continuous. The composition operators from A to B are precisely the unital homomorphisms from A to B. Composition operators satisfy $||C_\phi|| = 1 = C_\phi(1)$. If C_ϕ is a composition operator from A to A, then $C_\phi^n = C_{\phi^n}$, where $\phi^n = \phi \circ \cdots \circ \phi$ is the nth iterate of ϕ.

2. Essential spectral radius of composition operators

We denote the essential norm of an operator T from A to B by $||T||_e$. The essential norm is the distance from T to the compact operators from A to B,

$$||T||_e = \inf\{||T + K|| : K \text{ a compact operator}\}.$$

Theorem 1.1 leads to a brief proof of the following lemma. (See Theorems 8 and 9 of P. Gorkin and R. Mortini [GM]. See also [HIZ].)

Lemma 2.1. *Let C_ϕ and C_ψ be composition operators from A to B. If $\{y_j\}$ is a sequence in M_B such that $\{\phi(y_j)\}$ is not hyperbolically bounded in M_A, then*

$$||C_\phi - C_\psi||_e \;\geq\; \liminf \rho_A(\phi(y_j), \psi(y_j)).$$

Proof. Set $\liminf \rho_A(\phi(y_j), \psi(y_j)) = L$. Let K be a compact operator from A to B, and let $\varepsilon > 0$. Passing to a subsequence and applying Theorem 1.1 to $\{\phi(y_j)\}$, we can find a sequence of functions $\{F_k\}_{k=1}^{\infty}$ in A^{**} such that $F_k(\phi(y_j)) = 0$ for $j \neq k$, $F_k(\phi(y_k)) = 1$, and $\sum_{k=1}^{\infty} |F_k| \leq 1 + \varepsilon$ on $M_{A^{**}}$. Take $g_k \in A$ such that $||g_k|| < 1$, $g_k(\psi(y_k)) = 0$, and $g_k(\phi(y_k)) \to L$. Since $\sum |g_k F_k| \leq 1 + \varepsilon$, $g_k F_k \to 0$ weakly in A^{**}. As K^{**} is also compact,

$||K^{**}(g_kF_k)|| \to 0$. Consequently we obtain

$$\begin{aligned}(1+\varepsilon)||C_\phi - C_\psi + K|| &\geq \limsup ||(C_\phi^{**} - C_\psi^{**} + K^{**})(g_kF_k)|| \\ &= \limsup ||(C_\phi^{**} - C_\psi^{**})(g_kF_k)|| \geq \limsup |(C_\phi^{**}(g_kF_k))(y_k) - (C_\psi^{**}(g_kF_k))(y_k)| \\ &= \limsup |(g_kF_k)(\phi(y_k)) - (g_kF_k)(\psi(y_k))| = \lim |g_k(\phi(y_k))| = L.\end{aligned}$$

It follows that $||C_\phi - C_\psi||_e \geq L$, as required. □

Corollary 2.2. *Let C_ϕ be a composition operator from a uniform algebra A to a uniform algebra B. If $\phi(M_B)$ is not a hyperbolically bounded subset of M_A, then $||C_\phi||_e = 1$.*

Proof. Since $||C_\phi|| \leq 1$, also $||C_\phi||_e \leq 1$. For the reverse inequality, take ψ as any constant mapping with value, say, $a \in M_A$. Since $\phi(M_B)$ is not a hyperbolically bounded, there is sequence $\{\phi(y_j)\}$ such that $\lim \rho_A(\phi(y_j), a) = 1$. Clearly, C_ψ is a finite rank operator, so $||C_\phi||_e = ||C_\phi - C_\psi||_e \geq 1$ by the lemma above. □

We denote the essential spectrum of an operator T on A by $\sigma_e(T)$. Thus $\sigma_e(T)$ is the spectrum of the coset of T in the quotient Banach algebra of operators on A modulo compact operators. It is a compact subset of the complex plane, which is nonempty except in the trivial case that A is finite dimensional. We denote by $r_e(T)$ the essential spectral radius of T, that is, the supremum of $|\lambda|$ for $\lambda \in \sigma_e(T)$. By the spectral radius formula, $r_e(T) = \lim ||T^n||_e^{1/n}$.

Corollary 2.2 allows us to expand upon Theorem 3.3 of [GGL].

Theorem 2.3. *Let C_ϕ be a unital homomorphism of the uniform algebra A. The following statements are equivalent.*
(i) There is a decomposition of M_A into disjoint clopen subsets $E_1, \dots, E_m$ such that the iterates of ϕ converge uniformly on each E_j in the pseudohyperbolic metric to an attracting cycle in E_j for ϕ.
(ii) There is $n \geq 1$ such that the nth iterate $\phi^n(M_A)$ of M_A under ϕ is a hyperbolically bounded subset of M_A.
(iii) $r_e(C_\phi) < 1$, that is, the essential spectrum of C_ϕ does not contain any points of the unit circle.

Proof. The equivalence of (i) and (ii) is established in Theorem 3.3 of [GGL]. If (ii) fails, then the preceding lemma shows that $||C_\phi^n||_e = 1$ for all $n \geq 1$. By the spectral radius formula, $r_e(C_\phi) = 1$, and (iii) also fails. Thus (iii) implies (ii).

Suppose that (i) and (ii) hold. Replacing ϕ by some iterate ϕ^N, we can assume that the attracting cycles in (i) are attracting fixed points. Let A_0 be the subspace of functions in A that vanish at these fixed points. Then A_0 is an invariant subspace for C_ϕ. According to the proof of Theorem 3.3 in [GGL], there are $n \geq 1$ and $c < 1$ such that ϕ is a pseudohyperbolic contraction with contraction constant c on each of the sets $\phi^n(E_j)$. If $f \in A_0$, $x \in E_j$, and $y = \phi^n(x)$, then $|(C_\phi^{k+n}f)(x)| = |f(\phi^{k+n}(x))| = |f(\phi^k(y))| \leq \rho_A(\phi^k(y), x_j) \leq c^k \rho_A(y, x_j) \leq c^k$. Thus the norm of the restriction of C_ϕ^{n+k} to A_0 is bounded by c^k. Taking $(n+k)$th roots and sending k to ∞, we find that the spectral radius of the restriction of C_ϕ to A_0

is at most c. Since A_0 has finite codimension in A, the essential spectral radius of C_ϕ is at most c, and (iii) holds. □

A subset E of the spectrum M_B of a uniform algebra B is a *norming set* if

$$||f|| \;=\; \sup\{|f(x)| : x \in E\}, \qquad f \in B.$$

According to Lemma 3.8 of [GGL], if $\phi : M_B \to M_A$ corresponds to a composition operator $C_\phi : A \to B$, then $\phi(M_B)$ is hyperbolically bounded in M_A as soon as $\phi(E)$ is hyperbolically bounded in M_A. This leads to the following corollary.

Corollary 2.4. *Let B_X be the open unit ball of a Banach space X, and let A be a uniformly closed subalgebra of $H^\infty(B_X)$ that contains the functions in X^*. Let $\phi : B_X \mapsto B_X$ be an analytic map that induces a composition operator on A. Then $r_e(C_\phi) < 1$ if and only if there are $r < 1$ and an integer $n \geq 1$ such that $\phi^n(B_X) \subset rB_X$.*

Proof. Since B_X is a norming set for A, Lemma 3.8 of [GGL] shows that $\phi^n(M_A)$ is hyperbolically bounded if and only if $\phi^n(B_X)$ is hyperbolically bounded. Since $\rho_A(0,x) = ||x||$ for $x \in B_X$, this occurs if and only if $\phi^n(B_X)$ is contained in a ball rB_X for some $r < 1$. Thus the corollary follows from the equivalence of (ii) and (iii) in Theorem 2.3. □

If the Banach space X is finite dimensional, then either $r_e(C_\phi) = 1$ or $r_e(C_\phi) = 0$. Indeed, if $r_e(C_\phi) < 1$, then for n large, $\phi^n(B_X) \subseteq rB_X$, and consequently C_ϕ^n is a compact operator. It follows that $r_e(C_\phi) = 0$, and $\sigma_e(C_\phi) = \{0\}$. The same conclusion holds if A is a logmodular algebra, or more generally a URM-algebra (an algebra for which every $x \in M_A$ has a unique representing measure on the Shilov boundary; see [GL]).

If the Banach space X is infinite dimensional, it can occur that $0 < r_e(C_\phi) < 1$. Indeed, let λ be a complex number, $0 < |\lambda| < 1$, and define $\phi(x) = \lambda x$ for $x \in B_X$. The space $\mathcal{P}^k$ of homogeneous polynomials on X of degree k is an eigenspace for C_ϕ with eigenvalue λ^k. The eigenspace $\mathcal{P}^0$ is one-dimensional, consisting only of the constant functions, while $\mathcal{P}^1 = X^*$. Since the eigenspaces $\mathcal{P}^k$ are infinite dimensional for $k \geq 1$, the essential spectrum of C_ϕ includes λ^k for $k \geq 1$. Let us see that $\sigma(C_\phi) = \{0, 1, \lambda, \lambda^2, \cdots\}$. First, note that $\{\lambda^k : k = 0, 1 \ldots,\}$ are the only eigenvalues of C_ϕ. For $\mu \notin \{0, 1, \lambda, \lambda^2, \cdots\}$ choose $n \in \mathbb{N}$ such that $|\lambda|^n < |\mu|$ and let us check that the one to one map $\mu - C_\phi$ is an onto map. Consider the natural direct sum decomposition

$$H^\infty(B_X) \;=\; \mathcal{P}^0 \oplus \mathcal{P}^1 \oplus \cdots \oplus \mathcal{P}^n \oplus Q,$$

where $|f(x)| \leq ||f||\,||x||^{n+1}$ for $f \in Q$ and $x \in B_X$. For any $f \in Q$, the series $g := \mu^{-1} \sum_0^\infty (f \circ \phi^k)\mu^{-k}$ is uniformly convergent in $H^\infty(B_X)$ and $(\mu - C_\phi)(g) = f$. Moreover the subspace $\mathcal{P}^0 \oplus \mathcal{P}^1 \oplus \cdots \oplus \mathcal{P}^n$ is contained in the range of the map $\mu - C_\phi$ which thus turns to be an onto mapping. Since $1 \notin \sigma_e(C_\phi)$, we further obtain $\sigma_e(C_\phi) = \sigma(C_\phi) \backslash \{1\} = \{0, \lambda, \lambda^2, \cdots\}$, and $r_e(C_\phi) = |\lambda|$.

Consider now the one-dimensional case of the open unit disk $\mathbb{D}$ and a composition operator C_ϕ on $H^\infty(\mathbb{D})$ arising from an analytic map $\phi : \mathbb{D} \to \mathbb{D}$. As mentioned before, Zheng [Zh] proved that either $r_e(C_\phi) = 0$ or $r_e(C_\phi) = 1$, depending on whether there is $n \geq 1$ such that $\phi^n(\mathbb{D})$ is a relatively compact subset of $\mathbb{D}$. Further, Zheng showed that if $r_e(C_\phi) = 1$, and if ϕ has an attracting fixed point in $\mathbb{D}$, then the spectrum of C_ϕ fills out the closed unit disk $\overline{\mathbb{D}}$. It would be of interest to determine in a general setting exactly when $\sigma_e(C_\phi) = \overline{\mathbb{D}}$.

3. Hyperbolic vicinities of composition operators

We consider two composition operators C_ϕ and C_ψ from the uniform algebra A to a uniform algebra B. In analogy with the usual definition of Gleason parts, we say that C_ϕ and C_ψ are in the *same norm vicinity* if $||C_\phi - C_\psi|| < 2$, that is, if

$$\sup_{f \in \mathrm{ball}\, A,\ y \in M_B} |f(\phi(y)) - f(\psi(y))| \ = \ \sup_{y \in M_B} ||\phi(y) - \psi(y)|| \ < \ 2.$$

In view of the relation (4) between the norm and the pseudohyperbolic metric, this occurs if and only if there is $r < 1$ such that

$$\rho_A(\phi(y), \psi(y)) \leq r, \qquad y \in M_B.$$

As in the case of Gleason parts, being in the same vicinity is an equivalence relation. The norm vicinities form a partition of the set of composition operators from A to B into pairwise disjoint sets that are clopen with respect to the uniform operator norm.

We would like to develop a notion of vicinity that is adapted to the essential operator norm rather than the uniform operator norm. Towards this goal, we make the following definition.

Definition: Composition operators C_ϕ and C_ψ from A to B are in the *same hyperbolic vicinity* if there are a norming set Y for B, a subset E of Y, and an $r < 1$ such that

(i) $\rho_A(\phi(y), \psi(y)) \leq r$ for all $y \in E$, and

(ii) $\phi(Y \backslash E)$ and $\psi(Y \backslash E)$ are hyperbolically bounded in M_A.

Each norm vicinity is contained in a hyperbolic vicinity. We aim to show eventually that the hyperbolic vicinities partition the set of composition operators from A to B into pairwise disjoint sets that are clopen with respect to the essential operator norm. We begin with some lemmas.

Lemma 3.1. *Let $x_0 \in M_A$ and let $r < 1$. The pseudohyperbolic closed ball $B_r = \{x \in M_A : \rho_A(x, x_0) \leq r\}$ is a closed A-convex subset of M_A. Further, if $f_n \in A$ satisfy $||f_n|| < 1$ and $f_n(x_0) \to 1$, then $f_n \to 1$ uniformly on B_r.*

Proof. By (2), B_r is the set of $x \in M_A$ such that $|f(x)| \leq r$ for all $f \in A$ satisfying $||f|| < 1$ and $f(x_0) = 0$. The first statement follows immediately from this description. The second statement follows from the definition (1) of ρ_A and the fact that if the centers of pseudohyperbolic disks of radius r in $\mathbb{D}$ tend to 1, then the disks tend to 1. □

Lemma 3.2. *If the subset E of M_A is hyperbolically bounded, then the A-convex hull $\widehat{E}$ of E is hyperbolically bounded. Further, if $P_1, \ldots, P_m$ are distinct Gleason parts, and if $E = \cup_{j=1}^m E_j$ where E_j is a hyperbolically bounded subset of P_j, then $\widehat{E_j} \subset P_j$, and $\widehat{E} = \cup_{j=1}^m \widehat{E_j}$.*

Proof. Suppose E is contained in a union of closed pseudohyperbolic balls of the form $B_j = \{\rho_A(x, x_j) \le r_j\}$, where $x_j \in P_j$. By Lemma 3.1, $\widehat{E_j} \subset B_j$. Since x_1 and x_2 are in different Gleason parts, there are functions $f_n \in A$ satisfying $||f_n|| < 1$, $f_n(x_1) \to 1$, and $f_n(x_2) \to -1$. By Lemma 3.1, $f_n \to 1$ uniformly on B_1, and also $f_n \to -1$ uniformly on B_2. Thus $(1 + f_n)/2$ converges uniformly to 1 on B_1 and to 0 on B_2. By taking products of such functions constructed for the various x_j's, we see that the function that is 1 on B_k and 0 on the other B_j's can be approximated uniformly on $\cup B_j$ by functions in A. Thus the spectrum $\widehat{E} = \widehat{\cup E_j}$ of the uniform closure of the restriction of A to $\cup E_j$ coincides with $\cup \widehat{E_j}$. □

Lemma 3.3. *Let C_ϕ and C_ψ be composition operators from A to B, and let E be a subset of M_B. If $\rho_A(\phi(y), \psi(y)) \le r$ for $y \in E$, then $\rho_A(\phi(y), \psi(y)) \le r$ for $y \in \widehat{E}$.*

Proof. In view of the relation (4) between the norm and the pseudohyperbolic metric, it suffices to show that if $||\phi(y) - \psi(y)|| \le c$ for $y \in E$, then $||\phi(y) - \psi(y)|| \le c$ for $y \in \widehat{E}$. So suppose that $||\phi(y) - \psi(y)|| \le c$ for $y \in E$. Then $|f(\phi(y)) - f(\psi(y))| = |(C_\phi f)(y) - (C_\psi f)(y)| \le c$ for $f \in \operatorname{ball} A$ and $y \in E$. For each f, this estimate persists for $y \in \widehat{E}$. Consequently $||\phi(y) - \psi(y)|| \le c$ for $y \in \widehat{E}$. □

Theorem 3.4. *Let C_ϕ and C_ψ be composition operators from A to B that belong to the same hyperbolic vicinity, and let E be the set of $y \in M_B$ such that $\phi(y)$ and $\psi(y)$ belong to the same Gleason part of M_A. Then E is a clopen subset of M_B, and there is $r < 1$ such that $\rho_A(\phi(y), \psi(y)) \le r$ for $y \in E$. Further, M_B is the disjoint union of E and a finite number of disjoint clopen sets E_α such that for each index α, $\phi(E_\alpha)$ and $\psi(E_\alpha)$ are hyperbolically bounded subsets of different Gleason parts.*

Proof. Let Y and E be as in the definition of hyperbolic vicinity. Adjusting r if necessary, we can adjoin to E any subset S of M_B such that $\phi(S)$ and $\psi(S)$ are hyperbolically bounded subsets of the same part. Making a finite number of such adjunctions and adjustments, we can assume that E consists of all $y \in Y$ such that $\phi(y)$ and $\psi(y)$ belong to the same part.

Now $\phi(Y \backslash E) \cup \psi(Y \backslash E)$ is contained in a finite union of Gleason parts $P_1, \dots, P_m$ of M_A. For $j \ne k$, let E_{jk} be the set of $y \in Y$ such that $\phi(y) \in P_j$ and $\psi(y) \in P_k$. (Some of the E_{jk}'s may be empty.) Then Y is the disjoint union of E and the E_{jk}'s. By Lemma 3.2, $\phi(\widehat{E_{jk}})$ and $\psi(\widehat{E_{jk}})$ are hyperbolically bounded subsets of P_j and P_k respectively. By Lemma 3.3, $\rho_A(\phi(y), \psi(y)) \le r$ for $y \in \widehat{E}$. Thus $\widehat{E}$ and the $\widehat{E_{jk}}$'s are disjoint subsets of M_B.

Let $F(z, w) = (1 + z)(1 - w)/4$. Then $F(1, -1) = 1$, and $|F(z, w)| < 1$ for $|z| \le 1$, $|w| \le 1$, $(z, w) \ne (1, -1)$. There is a constant $c = c(r) < 1$ such that if $\zeta, \xi \in \mathbb{D}$ satisfy $\rho(\zeta, \xi) \le r$, then $|F(\zeta, \xi)| \le c$. Thus if $f \in A$ satisfies $||f|| < 1$, and if $y \in E$, then $|F(f(\phi(y)), f(\psi(y)))| \le c$. For any two pseudohyperbolic balls of radii < 1 in different Gleason parts, we can find a sequence of functions in the open unit ball of A that converges uniformly to 1 on one of the pseudohyperbolic balls and to -1 (or to 0) on the other. By combining such sequences, as in the proof of Lemma 3.2, we can find a sequence of functions $\{f_n\}$ in A such that $||f_n|| < 1$, $f_n \to 1$ uniformly on $\phi(E_{12}) \subset P_1$, $f_n \to -1$

uniformly on $\psi(E_{12}) \subset P_2$, and $f_n \to 0$ uniformly on subballs of P_j for $j \geq 3$. Set $g_n = F(f_n \circ \phi, f_n \circ \psi) \in B$. Then $||g_n|| < 1$, $g_n \to 1$ uniformly on E_{12}, $g_n \to 0$ uniformly on E_{21}, $g_n \to 1/4$ uniformly on the other E_{jk}'s, and $|g_n| \leq c$ on E. If $\{k_n\}$ is a sequence of integers that tends very slowly to $+\infty$, then $g_n^{k_n}$ still converges to 1 uniformly on E_{12}, while $g_n^{k_n} \to 0$ uniformly on the other E_{jk}'s and on E. In this way we see that the characteristic function of each E_{jk}, regarded as a function on Y, belongs to B. Since Y is a norming set for B, M_B is then the disjoint union of $\widehat{E}$ and the $\widehat{E_{jk}}$'s. □

The relation of belonging to the same hyperbolic vicinity is clearly reflexive and symmetric. This theorem shows that we may take as a norming set in the definition of hyperbolic vicinity the whole spectrum M_B and therefore, transitivity follows easily from it and König's inequality (3).

If M_B is connected, then only one of the sets among E and the E_α's can be nonempty, and we obtain the following.

Corollary 3.5. *Let C_ϕ and C_ψ be composition operators from A to B that belong to the same hyperbolic vicinity. Suppose that M_B is connected. Then either C_ϕ and C_ψ belong to the same norm vicinity, or $\phi(M_B)$ and $\psi(M_B)$ are hyperbolically bounded subsets of two different Gleason parts of M_A.*

Thus if M_B is connected, each hyperbolic vicinity is a norm vicinity, with one exception. The exception is the grand hyperbolic vicinity consisting of all composition operators C_ϕ from A to B such that $\phi(M_B)$ is hyperbolically bounded. If $B = H^\infty(\mathbb{D})$, or more generally if B is a URM-algebra with connected spectrum, then this grand hyperbolic vicinity coincides with the set of compact composition operators on B.

Corollary 3.6. *Suppose ϕ and ψ and analytic self-maps of some connected analytic set V. Then the induced composition operators C_ϕ and C_ψ on $H^\infty(V)$ are in the same norm vicinity as soon as they are in the same hyperbolic vicinity.*

Proof. Since V is connected, the spectrum of $H^\infty(V)$ is connected, and Corollary 3.5 applies. In this case, $\phi(V)$ and $\psi(V)$ are in the same Gleason part, namely that of V, so the second alternative of the corollary cannot occur. □

4. Subalgebras of finite codimension

We wish to consider the restrictions of homomorphisms from A to B to subalgebras A_s of A of finite (linear) codimension. By [Ga2], any such subalgebra can be embedded in a descending chain of subalgebras, $A = A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_m = A_s$, where each A_k has codimension 1 in A_{k-1}. Thus we need analyze only the case of a subalgebra A_s of codimension 1 in A.

So we assume that A_s is a subalgebra of A of codimension 1. According to [Ga2], there are two cases that can occur: either A_s is obtained from A by identifying two points of M_A, or A_s is the null space of a point derivation at some point of M_A. In the former case, the spectrum M_{A_s} is obtained from M_A by identifying the two points identified by A. In the latter case, M_{A_s} coincides with M_A. For $x \in M_A$, we denote by $\bar{x}$ the corresponding point of M_{A_s}, that is, the restriction of x to A_s. For a subset S of M_A, we denote by $\overline{S} = \{\bar{x} : x \in S\}$ the corresponding quotient subset of M_{A_s}.

We wish to relate the pseudohyperbolic metrics on M_A and M_{A_s}. We denote by ρ_s the pseudohyperbolic metric ρ_{A_s} on the spectrum M_{A_s} of A_s. Since $A_s \subset A$, we have

$$\rho_s(\bar{x},\bar{y}) \;\leq\; \rho_A(x,y), \qquad x,y \in M_A. \tag{5}$$

We need estimates in the other direction. To compare ρ_s and ρ_A, we divide the discussion into several cases.

Case 1. Suppose A_s consists of the functions $f \in A$ such that $f(x_0) = f(x_1)$, where x_0 and x_1 belong to distinct Gleason parts P_0 and P_1 of M_A. Thus $\overline{x_0} = \overline{x_1}$. Let $x \in M_A \backslash (P_0 \cup P_1)$, and let $y \in M_A$. We can find $f \in A$ such that $||f|| < 1$, $f(x_0) = f(x_1) = f(y) = 0$, and $f(x) \approx \rho_A(x,y)$. Since $f \in A_s$, we obtain $\rho_s(\bar{x},\bar{y}) \;\geq\; \rho_A(x,y)$. Thus

$$\rho_s(\bar{x},\bar{y}) \;=\; \rho_A(x,y) \tag{6}$$

for $x,y \in M_A$ providing $x \notin P_0 \cup P_1$. A similar argument shows that the identity (6) holds if both x and y belong to P_0, or if they both belong to P_1.

Suppose $x \in P_0$ and $y \in P_1$. Choose $f_0 \in A$ such that $||f_0|| < 1$, $f_0(x_0) = 0$, and $f_0(x) \approx \rho_A(x_0,x)$. Since y is in a different part from x, we can arrange that $f_0(y) \approx 1$. Choose $f_1 \in A$ such that $||f_1|| < 1$, $f_1(x_1) = 0$, $f_1(y) \approx -\rho_A(x_1,y)$, and $f_1(x) \approx 1$. Then $f = f_0 f_1 \in A_s$, $||f|| < 1$, $f(x) \approx \rho_A(x_0,x)$, and $f(y) \approx -\rho_A(x_1,y)$. The pseudohyperbolic distance from $f(x)$ to $f(y)$ in $\mathbb{D}$ is then close to $(\rho_A(x_0,x) + \rho_A(x_1,y))/(1 + \rho_A(x_0,x)\rho_A(x_1,y))$, and we obtain $\rho_s(\bar{x},\bar{y}) \geq (\rho_A(x_0,x) + \rho_A(x_1,y))/(1 + \rho_A(x_0,x)\rho_A(x_1,y))$. Since x_0 and x are in the same Gleason part, $\rho_A(x_0,x) = \rho_s(\overline{x_0},\bar{x})$, and also $\rho_A(x_1,y) = \rho_s(\overline{x_1},\bar{y})$. Thus from König's inequality (3), applied to ρ_s, we obtain equality, that is,

$$\rho_s(\bar{x},\bar{y}) \;=\; \frac{\rho_A(x_0,x) + \rho_A(x_1,y)}{1 + \rho_A(x_0,x)\rho_A(x_1,y)} \qquad x \in P_0,\ y \in P_1. \tag{7}$$

There is a similar identity if $x \in P_1$ and $y \in P_0$, and in every other case (6) holds. This determines completely the pseudohyperbolic metric ρ_s in terms of ρ_A. In particular, we see that the Gleason parts of A_s are the same as the Gleason parts of A, except that the two Gleason parts P_0 and P_1 of A are joined at $\overline{x_0} = \overline{x_1}$ to form a single part of A_s. Note that the identities connecting ρ_A and ρ_s show that a subset of M_A is hyperbolically bounded if and only if the corresponding quotient subset of M_{A_s} is hyperbolically bounded.

Case 2. Suppose A_s consists of the functions $f \in A$ such that $f(x_0) = f(x_1)$, where x_0 and x_1 belong to the same Gleason part P of M_A. As before, the identity (6) holds unless x and y both belong to P. Thus the Gleason parts of A_s are the same as those of A.

Suppose $x \in P$. Choose $f \in A_s$ such that $||f|| < 1$, $f(\overline{x_0}) = 0$, and $f(\bar{x}) \approx \rho_s(\overline{x_0},\bar{x})$. Since $f(x_0) = 0$, $|f(x)| \leq \rho_A(x_0,x)$, and since $f(x_1) = 0$, $|f(x)| \leq \rho_A(x_1,x)$. Hence

$$\rho_s(\overline{x_0},\bar{x}) \;\leq\; \min(\rho_A(x_0,x), \rho_A(x_1,x)), \qquad x \in P. \tag{8}$$

For a lower bound, choose $f_0 \in A$ such that $||f_0|| < 1$, $f_0(x_0) = 0$, and $f_0(x) \approx \rho_A(x_0,x)$, and similarly choose $f_1 \in A$ such that $||f_1|| < 1$, $f_1(x_1) = 0$, and $f_1(x) \approx \rho_A(x_1,x)$. Then

$f = f_0 f_1 \in A_s$, $f(\overline{x_0}) = 0$, and $f(\bar{x}) \approx \rho_A(x_0, x)\rho_A(x_1, x)$. It follows that

$$\min(\rho_A(x_0,x)^2, \rho_A(x_1,x)^2) \ \le\ \rho_A(x_0,x)\rho_A(x_1,x) \ \le\ \rho_s(\overline{x_0},\bar{x}), \qquad x \in P. \tag{9}$$

As before, these estimates show that a subset of M_A is hyperbolically bounded if and only if the corresponding quotient subset of M_{A_s} is hyperbolically bounded.

Continuing with f_0 and f_1, suppose that $y \in P$, and choose $g \in A$ such that $||g|| < 1$, $g(y) = 0$, and $g(x) \approx \rho_A(x, y)$. Then $h = gf_1f_2 \in A_s$, $||h|| < 1$, $h(y) = 0$, and $h(x) \approx \rho_A(x_0, x)\rho_A(x_1, x)\rho_A(x, y)$. Consequently

$$\rho_A(x_0,x)\rho_A(x_1,x)\rho_A(x,y) \ \le\ \rho_s(\bar{x},\bar{y}), \qquad x, y \in P. \tag{10}$$

This shows that as x tends to ∞ in the part P, the metrics $\rho_A(x, y)$ and $\rho_s(\bar{x}, \bar{y})$ are close to each other.

Case 3. Suppose that A_s is the null space of a point derivation at some point $x_0 \in M_A$, and let P denote the Gleason part containing x_0. Then A_s includes all products of the form fg, where $f, g \in A$ satisfy $f(x_0) = g(x_0) = 0$. Case 3 may be regarded as a limiting case of case 2 as $x_1 \to x_0$. If we proceed in analogy with case 2, with $x_0 = x_1$, we see that the Gleason parts of A_s are the same as those of A, and the identity (6) holds unless both x and y belong to P. The analogs of the estimates (9) and (10) , obtained in the same way, are

$$\rho_A(x_0,x)^2 \ \le\ \rho_s(\overline{x_0},\bar{x}), \qquad x \in P, \tag{11}$$

$$\rho_A(x_0,x)^2\rho_A(x,y) \ \le\ \rho_s(\bar{x},\bar{y}), \qquad x, y \in P. \tag{12}$$

Again the estimates show that a subset of M_A is hyperbolically bounded if and only if the corresponding quotient subset of M_{A_s} is hyperbolically bounded, and the metrics $\rho_A(x, y)$ and $\rho_s(\bar{x}, \bar{y})$ are close to each other as x tends to ∞ in the part P.

We summarize some of these results in the following theorem.

Theorem 4.1. *Let A_s be a unital subalgebra of finite codimension in the uniform algebra A. Then the spectrum M_{A_s} is obtained from M_A by identifying a finite number of pairs of points. A subset E of M_A is hyperbolically bounded in M_A if and only if the corresponding quotient subset of M_{A_s} is hyperbolically bounded in M_{A_s}.*

Proof. This follows from the corresponding results in each of the three cases treated above and the reduction of the general case to the case of codimension 1. □

Suppose now that C_ϕ and C_ψ are unital homomorphisms from the uniform algebra A to a uniform algebra B, corresponding to maps $\phi, \psi : M_B \to M_A$. Let A_s be a unital subalgebra of A of finite codimension. The restrictions of C_ϕ and C_ψ to A_s are unital homomorphisms from A_s to B, corresponding to maps $\bar{\phi}, \bar{\psi} : M_B \to M_{A_s}$ obtained respectively by following ϕ and ψ by the quotient map of M_A onto M_{A_s}, which simply identifies the (finite number of) pairs of points in M_A that are identified by A_s.

Theorem 4.2. *Let A and B be uniform algebras, and let A_s be a unital subalgebra of A of finite (linear) codimension. Let C_ϕ and C_ψ be composition operators from A to B, and* *let $C_{\bar{\phi}}$ and $C_{\bar{\psi}}$ be their restrictions to A_s. Then C_ϕ and C_ψ belong to the same hyperbolic vicinity if and only if $C_{\bar{\phi}}$ and $C_{\bar{\psi}}$ belong to the same hyperbolic vicinity.*

Proof. The estimate (4) and Theorem 4.1 show that if C_ϕ and C_ψ are in the same hyperbolic vicinity, then so are $C_{\bar{\phi}}$ and $C_{\bar{\psi}}$. We must show the converse. As before we assume that A_s is a subalgebra of A of codimension 1.

Assume that $C_{\bar{\phi}}$ and $C_{\bar{\psi}}$ are in the same hyperbolic vicinity. Let E denote the set of $y \in M_B$ such that $\overline{\phi(y)}$ and $\overline{\psi(y)}$ belong to the same Gleason part of M_{A_s}. Then $\overline{\phi(M_B\backslash E)}$ and $\overline{\psi(M_B\backslash E)}$ are hyperbolically bounded in M_{A_s}, so $\phi(M_B\backslash E)$ and $\psi(M_B\backslash E)$ are hyperbolically bounded in M_A. Now $\rho_s(\overline{\phi(y)}, \overline{\psi(y)}) = \rho_A(\phi(y), \psi(y))$ whenever $y \in E$ and $\phi(y)$, $\psi(y)$ are in some part other than P_0 or P_1 (in case 1) or P (in cases 2 or 3). The estimates (10) in case 2 and (12) in case 3 show that $\rho_A(\phi(y), \psi(y)) \leq r'$ for some $r' < 1$ and all $y \in E$ such that $\phi(y), \psi(y) \in P$, so that C_ϕ and C_ψ are in the same hyperbolic vicinity in cases 2 and 3.

So we assume we are in case 1, with the previous notation. In this case, if $\phi(y), \psi(y)$ both belong to the same Gleason part of M_A, then $\rho_s(\overline{\phi(y)}, \overline{\psi(y)}) = \rho_A(\phi(y), \psi(y))$. To complete the proof, it suffices to show that if E_0 is the set of $y \in E$ such that $\phi(y) \in P_0$ and $\psi(y) \in P_1$, and E_1 is the set of $y \in E$ such that $\phi(y) \in P_1$ and $\psi(y) \in P_0$, then $\phi(E_0)$, $\phi(E_1)$, $\psi(E_0)$, $\psi(E_1)$ are hyperbolically bounded in M_A. The identity (7) and the bound $\rho_s(\overline{\phi(y)}, \overline{\psi(y)}) \leq r$ show that $\rho_A(\phi(y), x_0) \leq r' < 1$ for $y \in E_0$, so that $\phi(E_0)$ and $\psi(E_0)$ are hyperbolically bounded. Similarly, $\phi(E_1)$ and $\psi(E_1)$ are hyperbolically bounded. □

Corollary 4.3. *Two composition operators C_ϕ and C_ψ from A to B belong to the same hyperbolic vicinity if and only if there is a unital subalgebra A_s of A of finite (linear) codimension such that the restrictions of C_ϕ and C_ψ to A_s belong to the same norm vicinity, that is, $||C_{\bar{\phi}} - C_{\bar{\psi}}|| < 2$.*

Proof. If $||C_{\bar{\phi}} - C_{\bar{\psi}}|| < 2$, the operators $C_{\bar{\phi}}$ and $C_{\bar{\psi}}$ are in the same hyperbolic vicinity with respect to A_s, hence by Theorem 4.2, C_ϕ and C_ψ are in the same hyperbolic vicinity. For the converse, suppose C_ϕ and C_ψ are in the same hyperbolic vicinity. Take clopen subsets E_α in M_B as in Theorem 3.4 so that $\phi(E_\alpha)$ and $\psi(E_\alpha)$ are hyperbolically bounded sets in different Gleason parts, say P_α and Q_α. For each α, select an arbitrary pair of points $x_\alpha \in P_\alpha$ and $x'_\alpha \in Q_\alpha$, and let A_s be the collection of functions $f \in A$ such that $f(x_\alpha) = f(x'_\alpha)$ for all α. Then A_s is a subalgebra of A of finite codimension. By Theorem 4.2, the restriction composition operators $C_{\bar{\phi}}$ and $C_{\bar{\psi}}$ are in the same hyperbolic vicinity. Since $\overline{\phi(y)}$ and $\overline{\psi(y)}$ belong to the same Gleason part of M_{A_s} for all $y \in M_B$, Theorem 3.4 shows that $C_{\bar{\phi}}$ and $C_{\bar{\psi}}$ belong to the same norm vicinity. □

5. Hyperbolic vicinities and essential operator norms

We are now ready to apply Lemma 2.1 to see that hyperbolic vicinities are open in the essential operator norm.

Theorem 5.1. *If the composition operators C_ϕ and C_ψ from A to B satisfy $||C_\phi - C_\psi||_e < 1$, then C_ϕ and C_ψ belong to the same hyperbolic vicinity.*

Proof. Let E be the set of $y \in M_B$ such that $\phi(y)$ and $\psi(y)$ belong to the same Gleason part of A. Suppose $\{y_j\}$ is a sequence in E such that $\rho_A(\phi(y_j), \psi(y_j)) \to 1$. If both $\{\phi(y_j)\}$ and $\{\psi(y_j)\}$ are hyperbolically bounded, then König's inequality (3) yields $\rho_A(\phi(y_j), \psi(y_j)) \leq s < 1$, contradicting our assumption. Thus we can assume that either $\{\phi(y_j)\}$ or $\{\psi(y_j)\}$ is not hyperbolically bounded. However, we now deduce from Lemma 2.1 that $||C_\phi - C_\psi||_e \geq 1$, in contradiction to our hypothesis. We conclude that there is $r < 1$ such that $\rho_A(\phi(y), \psi(y)) \leq r$ for all $y \in E$.

Now $\rho_A(\phi(y), \psi(y)) = 1$ for all $y \in M_B \backslash E$. Our hypothesis and Lemma 2.1 then show that $\{\phi(y_j)\}$ is hyperbolically bounded for any sequence in $M_B \backslash E$, and consequently $\phi(M_B \backslash E)$ is hyperbolically bounded, as is $\psi(M_B \backslash E)$. □

Corollary 5.2. *Hyperbolic vicinities of composition operators are open in the essential operator norm.*

Thus hyperbolic vicinities of composition operators form a partition of the set of composition operators from A to B into disjoint subsets that are clopen with respect to the essential operator norm.

Combining Corollary 5.2 and Corollary 3.5, we obtain the following.

Corollary 5.3. *If M_B is connected, and if C_ϕ is a composition operator from A to B such that $\phi(M_B)$ is not hyperbolically bounded, then the norm vicinity of C_ϕ is open with respect to the essential operator norm.*

6. Hyperbolic vicinity of the identity operator

In [HIZ], T. Hosokava, K. Izuchi, and D Zheng characterize the composition operators on $H^\infty(\mathbb{D})$ arising from analytic self-maps of $\mathbb{D}$ that are isolated with respect to the uniform operator norm. Further, they show that such operators are isolated with respect to the essential operator norm. In particular, the identity operator on $H^\infty(\mathbb{D})$ is isolated. It turns out that this result is quite general, as the following theorem shows.

Theorem 6.1. *A composition operator C_ψ on a uniform algebra A belongs to the same hyperbolic vicinity as the identity operator I on A if and only if $\psi(x) = x$ for all $x \in M_A$ with the possible exception of finitely many isolated points of M_A. In particular, if M_A has no isolated points, then the identity operator on A is isolated with respect to the essential operator norm in the set of composition operators on A.*

Proof. Note that $I = C_\phi$ for $\phi(x) = x$, $x \in M_A$. We consider the decomposition of M_A as a disjoint union of E and the (finitely many) sets E_α given by Theorem 3.4. Since E_α is clopen, it has a strong boundary point x_α. Since $\phi(E_\alpha)$ is contained in one Gleason part, and strong boundary points are one-point parts, we have $\phi(x) = x_\alpha$ for $x \in E_\alpha$, and E_α coincides with the singleton $\{x_\alpha\}$. Thus all the E_α's are singletons, which are isolated points of M_A.

On E we have $\rho_A(x, \psi(x)) \leq r < 1$. If $x \in E$ is a strong boundary point of A, then it comprises a one-point part, and consequently $\psi(x) = x$. Since the strong boundary points are dense in the Shilov boundary, this identity persists for all x in the Shilov boundary of the restriction algebra $A|_E$ of A to E. Consequently $C_\psi f = f$ on the Shilov boundary of the restriction algebra. It follows that $C_\psi f = f$ on the spectrum E of the restriction algebra. Thus $\psi(x) = x$ for $x \in E$. □

It was conjectured in [AGL] that the identity operator on $H^\infty(B_X)$ is isolated in the set of composition operators. The conjecture was settled affirmatively in [CHM]. Actually, the identity operator on any uniform algebra A is isolated in the set of composition operators on A with respect to the uniform operator norm. In fact, the identity operator forms by itself a norm vicinity. To see this, suppose that C_ψ belongs to the same norm vicinity as the identity operator. Then $\rho(x, \psi(x)) \leq r < 1$ for all $x \in M_A$, and we can take $E = M_A$ in the preceding proof. We conclude that $\psi(x) = x$ for all $x \in M_A$, and $C_\psi = I$.

A theorem analogous to Theorem 6.1 can be formulated for composition operators from A to B.

Theorem 6.2. *Let A and B be uniform algebras such that M_B is connected. Let C_ϕ be a composition operator from A to B. Suppose that ϕ maps strong boundary points in M_B to strong boundary points in M_A, and that ϕ is not constant. Then $\{C_\phi\}$ comprises by itself a hyperbolic vicinity. In particular, C_ϕ is isolated with respect to the essential operator norm in the set of composition operators from A to B.*

Proof. If C_ψ is in the same hyperbolic vicinity as C_ϕ, then in this case the sets E_α in Theorem 3.4 must be empty, and as before we obtain $\psi(y) = \phi(y)$ for strong boundary points, hence for all $y \in M_B$. □

References

[AGL] R. Aron, P. Galindo, and M. Lindström, *Connected components in the space of composition operators on analytic functions of many variables*, Int. Eq. Op. Theory **45** (2003), 1-14.

[CHM] C. H. Chu, R. V. Hügli, and M. Mackey, *The identity is an isolated composition operator in $H^\infty(B)$*, (preprint 2003).

[Da] A. Davie, *Linear extension operators for spaces and algebras of functions*, Amer. J. Math. **94** (1972), 156-172.

[GGL] P. Galindo, T. W. Gamelin, and M. Lindström, *Composition operators on uniform algebras and the pseudohyperbolic metric*, J. Korean Math. Soc. **41** (2004), 1-20.

[GL] P. Galindo and M. Lindström, *Factorization of homomorphisms through $H^\infty(D)$*, J. of Math. Anal. and Appl. **280** (2003), 375-386.

[Ga1] T. Gamelin, *Uniform Algebras*, second edition, AMS Chelsea Publishing, 1984.

[Ga2] T. Gamelin, *Embedding Riemann surfaces in maximal ideal spaces*, J. Functional Anal. **2** (1968), 123-146.

[Ga3] T. Gamelin, *Uniform algebras on plane sets*, in *Approximation Theory*, edited by G.G. Lorentz, Academic Press, (1973), 101–149.

[Ga4] T. Gamelin, *Homomorphisms of uniform algebras*, in *Recent Progress in Functional Analysis*, edited by Bierstedt et al. , North-Holland Elsevier, 2001, 95-105.

[GM] P. Gorkin and R. Mortini, *Norms and essential norms of linear combinations of endomorphisms*, to appear.

[HIZ] T. Hosokava, K. Izuchi, and D. Zheng, *Isolated points and essential components of composition operators on H^∞ functions*, Proc. Amer. Math. Soc. **130** (2002), 1765-1773.

[Ka] H. Kamowitz, *Compact endomorphisms of Banach algebras*, Pac. J. Math. **89** (1980), 313–325.

[Kl] U. Klein, *Kompakte multiplikative Operatoren auf uniformen Algebren*, Dissertation, Karlsruhe, 1996.

[Kö] H. König, *Zur abstrakten Theorie der analytischen Funktionen II*, Math. Ann. **163** (1966), 9–17.

[Zh] L. Zheng, *The essential norms and spectra of composition operators on H^∞*, Pacific J. Math. **203** (2002), 503–510.

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