Collection of Errata, Clarifications, and Comments
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COMPLEX DYNAMICS
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Second printing: typos and minor changes

p.4, l.-10: Change “ $f(\partial\Delta)$ ” to “ $\partial f(\Delta)$ ”

p.4, l.-2: Change “ are ” to “ is ”

p.7, l.3: Change “ w_n ” to “ w_0 ”

pp.9-10: Change the first sentence in Section 3 to read: “ It was P. Montel (1911) who first formulated the notion of a normal family of meromorphic functions and proved the criterion that bears his name. ” Montel's original proof was based on Schottky's theorem, not on Picard's modular function.

p.13, l.-4: Change “ the the ” to “ the ”

p.14, l.6: Change “ 2.1 ” to “ 3.1 ”

p.16, Fig.3: Add horizontal bars to the fractions $\frac{\pi}{2}$ and $\frac{\arg\,\mu(z)}{2}$

p.19, l.-7: Insert “ has nonvanishing Jacobian ” after “ If $f \in QC^1(k, R)$ ”

p.20, l.-9: Change “ $||U_\mu||$ ” to “ $||U_\mu||_p$ ” (insert subscript p)

p.20, l.-7: Change “ $||(I - U_\mu)^{-1}||$ ” to “ $||(I - U_\mu)^{-1}||_p$ ”(insert subscript p)

p.22, l.-5: Change “ the smoothness of f ” to “ $f \in QC^1(k, R)$ ”

p.22, l.-2: Change “ If f were C^1 we would ” to “ If f were C^1 and $f_z \neq 0$, we would ”

p.30, l.-8: Change “ due in this form ” to “ due essentially in this form ”

p.33, l.-13: Change “ For $|z|$ small there is ” to “ Choose ”

p.33, l.-12: Change “ $C|z|^p$. ” to
“ $C|z|^p$ for $|z| \leq 1/C$. Then $|f(z)| \leq |z|$ for $|z| \leq 1/C$. ”

p.33, l.-10: Change “ δ ” to “ $1/C$ ”

p.33, lines 1 to 2: Delete “ at the origin ”

p.34, l.7: Delete “ $c <$ ” so that it reads “ $|z| \leq 1/C$ ”

p.36, l.-1: Change comma to period after the last estimate, and add the line:
“ where the estimate is uniform for z belonging to a compact set. ”

p.39, l.9: Change “ to to ” to “ to ”

p.41, l.-14: Change “ let ” to “ suppose $z_0 = 0$ is a fixed point of $f(z)$, with multiplier ” p.71, l.-13: Change “ inside ” to “ on the bounded components of ”

p.71, l.-3: Change “ $2d-1$ ” to “ $2d+1$ ”

p.71, l.-2: Change “ $2d-1$ ” to “ $2d+1$ ”

p.74, l.-1: Change “ an isometry ” to “ a local isometry ”

p.75, l.3: Change “ an isometry ” to “ a local isometry ”

p.75, l.4: Change “ $z, w \in U$. In particular for any ” to “ $z, w \in U, z \neq w$. Further, for any ”

p.75, l.18: Change “ an isometry ” to “ a local isometry ”

p.75, l.-5: Change “ By an isometry, we mean at the local level, so that the lift ” to “ Since f is a local isometry, the lift ”

p.76, l.10: After the first sentence, insert “ We claim that either (1) or (2) holds. For this, suppose that (2) fails. ”

p.77, l.15: Change “ (2) ” to “ (1) ”

p.87, l.2: Change “ λ^{n-1} ” to “ λ^n ”

p.91, l.14: Change “ had been ” to “ is ”

p.91, l.15-16: Delete the sentence “ Recently … $z^{16}+c$. ”

p.100, l.3: Change “ through ” to “ around ”

p.101, l.10: Change “ conjugate to ” to “ conjugate on U_1 to ”

p.128, l.15: Change “ are dense in $\mathcal{M}$ ” to “ are dense in $\partial\mathcal{M}$ ”

p.149, format: Insert space at end of example, between lines -5 and -6

p.154, l.-2: Insert “ $= f(z,c)$ ” after “ $P_c^\ell(z)$ ”

p.157, format: Insert space at end of statement of theorem, between lines 16 and 17

p.173, Index: Change “ repulsive cycle, ?? ” to “ repulsive cycle, 172 ” page 11, proof of Theorem I.3.2 (Montel's theorem)

The very last assertion of the proof requires justification. To do this, we follow the proof given, except that we take ψ to be the universal covering map of the upper half-plane $\mathbb{H}$ over $\mathbb{C}\backslash\{0,1\}$ constructed in the proof of Theorem 3.1, with fundamental domain E from that proof, and we choose the lifts $\tilde{f}$ of functions $f \in \mathcal{F}$ so that $\tilde{f}(0) \in E$. The functions $\tilde{f}$ still form a normal family. Let $\{f_n\}$ be a sequence in $\mathcal{F}$. Passing to a subsequence, we can assume that $\tilde{f}_n$ converges normally to g on $\mathbb{H}$. We must show that f_n converges normally. If $\tilde{f}_n(0)$ converges to a point of $\mathbb{H}$, then the image of g is in $\mathbb{H}$, and we can use the local inverses of ψ to see that f_n converges normally to the analytic function $\psi \circ g$. Our problem is to determine what happens when the limit of $\tilde{f}_n(0)$ does not belong to $\mathbb{H}$.

If the limit of $\tilde{f}_n(0)$ is not in $\mathbb{H}$, then since $\tilde{f}_n(0) \in E$, either $\operatorname{Re} \tilde{f}_n(0) \to +\infty$, or $\tilde{f}_n(0)$ converges to one of the corners 0 or 1 of ∂E. By composing the functions in the family $\mathcal{F}$ with a fractional linear transformation that permutes the points 0, 1, ∞, we can assume that $\operatorname{Re} \tilde{f}_n(0) \to +\infty$. Then by Harnack's theorem, $\operatorname{Re} \tilde{f}_n(z) \to +\infty$ uniformly on compacta. Using the periodicity of ψ, we see that $|\psi(w)| \to \infty$ uniformly as $\operatorname{Re} w \to +\infty$. Hence $f_n = \psi \circ \tilde{f}_n$ converges to ∞ uniformly on compacta, and in particular it converges normally, as required.

A variant of the proof, which avoids Harnack's theorem, proceeds in outline as follows. Replacing the family of functions $\mathcal{F}$ by the family of their square roots, one assumes that the family omits four points $\{-1, 0, 1, \infty\}$ in the extended plane. Then one proceeds as above, to the case where $\tilde{f}_n(0)$ converges to a vertex of E. In this case one considers the compositions $g_n = \varphi \circ f_n$, where φ is the fractional linear transformation that maps -1 to that vertex and leaves the other two vertices of E fixed. Now $\tilde{g}_n(0) = \varphi(\tilde{f}_n(0))$ converges to a point of $\mathbb{H}$. We conclude as before that g_n converges normally, as does f_n.

page 55, proof of Theorem III.1.1

Theorem 1.1 requires some justification to the effect that a neutral fixed point in the Fatou set belongs to a Siegel disk as defined. The following lemma clarifies the definition of a Siegel disk, and Theorem 1.1 follows immediately.

Lemma *Let 0 be a neutral fixed point for a rational function R, with multiplier λ. If $0 \in \mathcal{F}$, and if U is the component of the Fatou set containing 0, then Schröder's equation $\varphi(R(z)) = \lambda\varphi(z)$, with side conditions $\varphi(0) = 0$, $\varphi'(0) = 1$, has a (unique) solution $\varphi(z)$ defined on U and mapping U conformally onto a disk.*

Proof. We can assume that $\infty \in \mathcal{J}$. Note that $R(U) \subseteq U$. Since the iterates R^n form a normal family on U, they are uniformly bounded on compact subsets of U. As in the proof of Theorem II.6.2, the functions $\varphi_n(z) = (1/n)\sum_{j=0}^{n-1} \lambda^{-j} R^j(z)$ are uniformly bounded on compact subsets of U, and any limit $\varphi(z)$ of the φ_n's has the required properties. $\square$

page 77, proof of Lemma IV.2.3

To see that one of the alternatives (1), (4), or (5) holds, proceed as follows.

Suppose that U is a punctured disk, say $U = \Delta\backslash\{0\}$, with covering map $\psi(\zeta) = e^{2\pi i\zeta}$ from the upper half-plane $\mathbb{H}$ to U. If f is a local hyperbolic isometry of U, and F is the lift of f to $\mathbb{H}$, then $F(\zeta+1) \equiv F(\zeta)$, so there is an integer m such that $F(\zeta+1) = F(\zeta)+m$. Thus F fixes ∞, and F is affine. Evidently $F(\zeta) = m\zeta + b$ where $m \geq 1$ and b is real. Thus $f(z) = e^{2\pi ib}z^m$. If $m > 1$, then (1) holds. If $m = 1$, then since Γ is discrete, b is irrational, and (5) holds.

A similar argument shows that if U is an annulus, then (4) holds.

page 90, proof of Theorem V.2.3

In this proof, the a and c do not come directly from the statement of Lemma 2.1. They come from an open neighborhood V of $\mathcal{J}$, as follows. Let V be an ε-neighborhood of $\mathcal{J}$ with respect to the hyperbolic metric of $D = \overline{\mathbb{C}}\backslash CL$. Then $R^{-1}(V) \subset V$. For $\varepsilon > 0$ small, there is $A > 1$ such that (2.1) holds for $z \in V$. Set $c = 1/A$ and $a = (\sup\sigma)/(\inf\sigma)$, where the sup and the inf are taken over V. Then $|(R^k)'(z)| \geq a/c^k$ for all $z \in V$ such that $R^k(z) \in V$, as in the proof of Lemma 2.1.

page 143, proof of Theorem VIII.5.2

There is a gap in the proof, which requires substantial work to fill. The problem is to show that if P_a has a parabolic cycle, then θ has odd denominator. The gap is filled, and in a more general setting, in the Doctoral Dissertation of Gustav Ryd, "Iterations of one parameter families of complex polynomials," Department of Mathematics, KTH, Stockholm (1997), ISBN 91-7170-210-5. The relevant statement is Proposition 5.8, whose proof covers pages 38-43.

Ryd's thesis contains much more. In particular, it contains (Dissertation Section 3) theorems on the landing of external rays at parabolic and repelling periodic points of the Julia set of a rational function. It also carries out (Dissertation Section 7 and Theorem 8.1) the "main deformation construction" sketched at the end of Section VIII.7, again in a more general setting.

Ryd devotes special attention to one-parameter families of polynomials that have the form

$$P_c(z) = z^d + \alpha_{d-1}(c)z^{d-1} + \ldots + \alpha_0(c), \qquad P_c'(z) = d\prod_{j=1}^{d-1}(z - p_j(c)),$$

where $\alpha_0(c), \ldots, \alpha_{d-1}(c)$ and $p_1(c), \ldots, p_{d-1}(c)$ are polynomials in c. This includes such one-parameter families such as $z^d + c$, and more generally $p(z) + c$, where p is a polynomial. Thus each critical point has polynomial dependence on c, and one can define a "Mandelbrot set" $\mathcal{M}_j$ for each critical point. Ryd investigates the behavior of P_c as $c \to a \in \mathcal{M}_j$.