CORTONA SUMMER COURSE NOTES

SUMMARY: These notes include (1) a list below of ancillary references, (2) a description of various proof techniques for the basic conjugation theorems, and (3) a page of miscellaneous comments and alternate proofs of several of the theorems in the Complex Dynamics book.

The notes were prepared in connection with a summer course in holomorphic dynamics, sponsored by the Scuola Matematica Interuniversitaria at Cortona, Italy, July-August, 2003. The main organizer was Stefano Marmi of the University of Pisa. There were two series of lectures, one organized by myself on foundational material, the other by Marmi on small divisor problems. The lecture series were based respectively on the following books:

Lennart Carleson and Theodore W. Gamelin, *Complex Dynamics*, Springer-Verlag, 1993.

Stefano Marmi, *An Introduction to Small Divisors Problems*, Dipartimento di Matematica, Università di Pisa, 2002.

There were a number of presentations by participants based on other materials. Participants spent some time exploring Julia sets and the Mandelbrot set on the web.

Miscellaneous references

Alan F. Beardon, *Iteration of Rational Functions*, Springer-Verlag, 1991.

Norbert Steinmetz, *Rational Iteration: Complex Analytic Dynamical Systems*, de Gruyter, 1993.

John Milnor, *Dynamics in One Complex Variable, Introductory Lectures*, 2nd Edition, Vieweg, 2000; reviewed by John Hubbard, Bull. Amer. Math. Soc. 38 (2001), 495-498.

Daniel S. Alexander, *A History of Complex Dynamics from Schröder to Fatou and Julia*, Vieweg, 1994; reviewed by T.W.Gamelin, Historia Math 23 (1996), 74-83.

Lars V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill, 1973.

Theodore W. Gamelin, *Complex Analysis*, Springer-Verlag, 2001.

T.W.Gamelin, *Conjugation theorems via Neumann series*, MRS Report 99-1 (1999), Math Department, UCLA.

Miscellaneous web sites

www.math.bu.edu/DYSYS/: Robert Devaney's web site for fractals.

www.math.bu.edu/DYSYS/explorer/page1.html: exploring Mandelbrot and Julia sets.

www.math.ucla.edu/∼twg/CD.book.html: web site for Complex Dynamics book. It includes a link to the technical report listed above.

Five Proofs of the Theorem of Koenigs

There are several proofs in circulation for the various conjugation theorems. To provide a focus for the proof techniques, we indicate five proofs of Koenigs' theorem.

1. Proof by contraction mappings

The proofs of the theorems of Koenigs and Boettcher given in the text (Theorems II.2.1 and II.4.1) are perhaps the shortest and most direct. They can be recast as applications of the contraction mapping principle to mappings of spaces of analytic functions. The KAM technique used in the proof of Siegel's theorem (Theorem II.6.4) is a very refined extension of the contraction mapping principle.

2. Proof by composition operators

Another proof method, which is related to the contraction mapping principle, involves linearizing the problem in function space and converting it to the operator equation

$$(\lambda I - T)\varphi = \psi.$$

Here T is a composition operator, $T\varphi = \varphi \circ f$, on an appropriate space of analytic functions. The solution is given by a Neumann series

$$\varphi = (\lambda I - T)^{-1}\psi = \sum_{n=0}^{\infty} \lambda^{-(n+1)} T^n \psi.$$

In the case of Boettcher's theorem, the relevant composition operator is quasi-nilpotent. In the case of Koenigs' theorem, the Koenigs function appears as the eigenfunction of the composition operator corresponding to the largest nontrivial eigenvalue. For full details, see the technical report *Conjugation theorems via Neumann series* referenced above.

The Leau-Fatou theorem can be proved by roughly the same method, though more work is required to obtain convergence. In this case, the relevant equation is $(I - T)\varphi = \psi$, and the solution is given by $\varphi = \sum T^n \psi$. A detailed proof is contained in the technical report just mentioned.

3. Proof by the method of majorants

The method of majorants involves finding a formal solution of the equation, and then comparing the coefficients of the formal solution with those of a known convergent series to obtain convergence. The method of majorants was used by Siegel in his original 1942 proof of Theorem II.6.4. To illustrate the method, we use it to prove Koenigs' theorem.

Suppose $0 < |\lambda| < 1$, and suppose

$$f(z) = \lambda z + \sum_{n=2}^{\infty} a_n z^n$$

is analytic near 0. Koenigs' theorem boils down to finding an analytic function

$$\psi(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (1)$$

that satisfies the "Poincaré equation"

$$f(\psi(z)) = \psi(\lambda z), \qquad |z| < \varepsilon. \tag{2}$$

The inverse $\varphi = \psi^{-1}$ of ψ then satisfies Schröder's equation $\varphi(f(z)) = \lambda\varphi(z)$.

Now the formal power series (1) for $\psi(z)$ satisfies (2) if and only if the b_n's satisfy

$$b_n = \frac{1}{\lambda^n - \lambda} \sum_{k=2}^{n} a_k \Big(\sum_{n_1+\cdots+n_k=n} b_{n_1} \cdots b_{n_k} \Big), \qquad n \geq 2, \tag{3}$$

where $n_j \geq 1$ for $1 \leq j \leq k$, and $b_1 = 1$. We must show that if the coefficients b_n are defined recursively by (3), the resulting series (1) for $\psi(z)$ has positive radius of convergence.

As an auxiliary function we use the analytic solution $h(z)$ of the equation

$$h(z) = z + \frac{h(z)^2}{1 - h(z)}, \qquad h(0) = 0. \tag{4}$$

We mention that $h(z) = (1 + z + \sqrt{1 - 6z + z^2})/4$, though we make no use of this explicit expression. Let

$$h(z) = \sum_{k=1}^{\infty} M_k z^k.$$

From (4) we have

$$h(z) = z + \sum_{k=2}^{\infty} h(z)^k.$$

If we substitute the power series for $h(z)$ into this expression and equate coefficients, we obtain $M_1 = 1$, $M_2 = M_1^2 = 1$, $M_3 = M_1M_2 + M_2M_1 + M_1^3 = 3$, and more generally

$$M_n = \sum_{k=2}^{n} \Big(\sum_{n_1+\cdots+n_k=n} M_{n_1} \cdots M_{n_k} \Big), \qquad n \geq 2, \tag{5}$$

where again $n_j \geq 1$ for $1 \leq j \leq k$. Since $\sum a_k z^k$ has positive radius of convergence, there are $C_0, R \geq 1$ such that $|a_n| \leq C_0 R^{n-1}$ for $n \geq 1$. If we set $C = 1/|\lambda|(1-|\lambda|)$, then $1/|\lambda^n - \lambda| \leq C$ for $n \geq 1$, and we obtain by induction from (3) and (5) that $|b_n| \leq (C_0RC)^{n-1}M_n$. Indeed, using the induction hypothesis, we estimate

$$\begin{aligned} |b_n| &\leq C \sum_{k=2}^{n} C_0 R^{k-1} \Big(\sum_{n_1+\cdots+n_k=n} |b_{n_1}| \cdots |b_{n_k}| \Big) \\ &\leq CC_0 \sum_{k=2}^{n} R^{k-1} \Big(\sum_{n_1+\cdots+n_k=n} (C_0RC)^{n-k} M_{n_1} \cdots M_{n_k} \Big) \leq (C_0RC)^{n-1} M_n. \end{aligned}$$

Consequently the series defining $\psi(z)$ has a positive radius of convergence, and the conjugation exists.

For applications of this method to the Siegel-Brjuno theorem and Yoccoz's theorem, see Marmi's book referenced above.

4. Proof by quasiconformal mapping

The proof of the Leau-Fatou theorem described in Section II.5, based on quasiconformal mapping, carries over to a proof of Koenigs' theorem, and the details simplify.

Suppose that $f(z) = \lambda z + O(|z|^2)$, where $0 < |\lambda| < 1$, and suppose for convenience that $|f(z) - \lambda z| \leq \varepsilon |z|^2$ for $|z| \leq 1$. Let $T = \{|z| = 1\}$ denote the unit circle. Then $f(T)$ is close to the circle $\{|z| = |\lambda|\}$.

Fix $0 < \rho < 1$, and denote the dilation $\zeta \mapsto \rho\zeta$ by G,

$$G(\zeta) = \rho\zeta, \qquad \zeta \in \mathbf{C}.$$

Let $z = h(\zeta)$ be a smooth diffeomorphism of the genuine annulus $\{\rho \leq |\zeta| \leq 1\}$ and the annular domain between T and $f(T)$ such that $h(\zeta) = \zeta$ for $|\zeta| = 1$, and h maps the circle ρT to $f(T)$ by $h(\rho\zeta) = f(\zeta)$ for $|\zeta| = 1$. We assume that the dilatation of h is bounded by $k < 1$, so that h is quasiconformal. (To use the precise result established in Section I.6, we arrange for $h(\zeta)$ to be analytic on a small disk.) Now the forward iterates of T under f divide the punctured disk $\Delta \backslash \{0\}$ into annular domains, each the image of the annular domain between T and $f(T)$ under exactly one iterate of f. Consequently we can extend h to the punctured disk so that $h \circ G = f \circ h$, that is,

$$h(\rho\zeta) = f(h(\zeta)), \qquad 0 < |\zeta| \leq 1.$$

Consider the ellipse field corresponding to h on $\Delta \backslash \{0\}$, assigning to ζ the infinitesimal ellipse at ζ that is mapped to an infinitesimal circle by h. Let $\mu(\zeta)$ be the corresponding Beltrami coefficient. By interpreting in terms of infinitesimal ellipses, or by referring to the formulae for pre- and post-composition of μ with analytic functions (Section I.5), we see that $\mu(\zeta)$ is invariant under G, that is,

$$\mu(G(\zeta)) = \mu(\zeta)$$

for $0 < |\zeta| \leq 1$. We extend μ to $\mathbf{C} \backslash \{0\}$ so that this identity holds for all $\zeta \neq 0$. (Extending μ to forward orbits requires special circumstances; extending μ to backward orbits is easy.) Note that the extended μ also satisfies $|\mu| \leq k < 1$, so we can solve the Beltrami equation. Let ψ denote the solution to the corresponding Beltrami equation, normalized so that $\psi(0) = 0$ and $\psi(\infty) = \infty$. Since μ is invariant under G, $\psi \circ G \circ \psi^{-1}$ maps infinitesimal circles to infinitesimal circles. Hence $\psi \circ G \circ \psi^{-1}$ is analytic, a fractional linear transformation, and $(\psi \circ G \circ \psi^{-1})(\xi) = \alpha\xi$ for some $\alpha \neq 0$. Thus $\psi(G(\zeta)) = \alpha\psi(\zeta)$ for all $\zeta \in \mathbf{C}$.

We define $\varphi(z) = (\psi \circ h^{-1})(z)$ for $0 < |z| < 1$. Then $\varphi(z)$ maps infinitesimal circles to infinitesimal circles, so $\varphi(z)$ is analytic for $0 < |z| < 1$, and since $\varphi(z) \to 0$ as $z \to 0$, $\varphi(z)$ is also analytic at $z = 0$. Now $\varphi \circ f = \psi \circ h^{-1} \circ f = \psi \circ h^{-1} \circ f \circ h \circ h^{-1} = \psi \circ h^{-1} \circ h \circ G \circ h^{-1} = \psi \circ G \circ h^{-1} = \psi \circ G \circ \psi^{-1} \circ \psi \circ h^{-1} = \alpha\psi \circ h^{-1} = \alpha\varphi$. Thus φ is an analytic conjugation of f to multiplication by α. It follows that $\alpha = \lambda$, and φ is a solution of Schröder's equation.

5. Proof by uniformization

We assume we are in the same situation as above, so T is the unit circle, and $f(T)$ is close to the circle $\{|z| = |\lambda|\}$. We form a Riemann surface R from the annular domain between T and $f(T)$ by identifying $z \in T$ to $f(z)$. The surface R can be viewed as the quotient space obtained from $\Delta\backslash\{0\}$ by identifying z and $f(z)$ for all $z \in \Delta\backslash\{0\}$. The corresponding quotient map $\Delta\backslash\{0\} \to R$ is a covering map.

The surface R is topologically a torus. By the uniformization theorem, there is a discrete lattice $\mathcal{L}$ in $\mathbf{C}$ such that R is conformally equivalent to $\mathbf{C}/\mathcal{L}$. We may assume that the universal covering map of $\mathbf{C}$ onto R maps $0 \in \mathbf{C}$ to $1 \in R$. We may assume also that $\mathcal{L}$ is generated by 1 and a lattice point τ in the upper half-plane, and that these generators are chosen so that the loop $t \mapsto e^{2\pi i t}$, $0 \le t \le 1$, lifts to a path in $\mathbf{C}$ from 0 to 1, and so that some fixed path across the annulus from 1 to $f(1)$ lifts to a path in $\mathbf{C}$ from 0 to τ.

Now we define a map φ from the punctured disk $\overline{\Delta}\backslash\{0\}$ to $\mathbf{C}\backslash\{0\}$ as follows. We choose a path γ in the punctured disk from 1 to z, we lift γ to a path $\tilde{\gamma}$ in $\mathbf{C}$ starting at 0, and we set $w = \varphi(z) = e^{2\pi i\zeta}$, where ζ is the endpoint of $\tilde{\gamma}$. If γ_0 is another path in the punctured disk from 1 to z, then $\gamma_0^{-1}\gamma$ is homotopic in the punctured disk to a path consisting of a finite number of loops around the unit circle, so the terminal point of the lift of γ_0 is congruent modulo the integers to the terminal point ζ of $\tilde{\gamma}$, and the values w coincide. Thus $\varphi(z)$ is well-defined, and $\varphi(z)$ is analytic. Since the lift of a path from z to $f(z)$ is a path from ζ to a point congruent to $\zeta + \tau$ modulo the integers, we see that $\varphi(f(z)) = e^{2\pi i(\zeta+\tau)}$. Setting $\lambda = e^{2\pi i\tau}$, we thus have $\varphi(f(z)) = \lambda\varphi(z)$. By Riemann's theorem on removable singularities, $\varphi(z)$ extends to be analytic at $z = 0$, and so φ is a solution of Schröder's equation.

The same line of proof serves to establish the Leau-Fatou theorem. In this situation we assume that $g(z)$ is analytic and univalent in the right half-plane, and that $g(z) - (z + 1)$ is small in some sense. We form the Riemann surface R from the strip-like domain D between the imaginary axis $i\mathbf{R}$ and $g(i\mathbf{R})$ by identifying the points iy and $g(iy)$, and we assume that the iterates under g of D and its boundary fill out the right half-plane. From the uniformization theorem we see that R is the punctured complex plane, a punctured disk, or an annulus, depending on whether the two ends of R corresponding to $iy \to \pm\infty$ are "point punctures" or "closed analytic curves." Elementary estimates with extremal length (Chapter 4 of the Ahlfors book Conformal Invariants) show that the module of the annular part of the surface corresponding to the part of D between a segment from 0 to $g(0)$ and a segment from iM to $g(iM)$ tends to ∞ as $M \to +\infty$. It follows that R has a removable singularity (puncture) at $i\infty$, and R also has a removable singularity at $-i\infty$. Thus R is conformally equivalent to the punctured complex plane $\mathbf{C}\backslash\{0\}$. The surface R can be viewed as the quotient space obtained from the right half-plane by identifying z and $g(z)$ for all z. Let χ be the corresponding quotient map. We use the universal covering map $\zeta \mapsto e^{2\pi i\zeta}$ of $\mathbf{C}$ onto $\mathbf{C}\backslash\{0\}$ to lift the map χ to a map φ from the right half-plane to $\mathbf{C}$. The lift satisfies "Abel's equation" $\varphi(g(z)) = \varphi(z) + 1$. Thus the function $\varphi(z)$ obtained in this way is the "Fatou coordinate function" on the right half-plane.

Miscellaneous comments and alternate proofs

page 3, Theorem I.1.4

The "4" is not necessary for the upper bound. In fact, we have the sharp estimate

$$\mathrm{dist}(f(z_0), \partial(f(D))) \ \le\ |f'(z_0)|\mathrm{dist}(z_0, \partial D).$$

This follows by scaling from the following lemma.

Lemma *If $f \in S$, and f is not the coordinate function z, then $\partial f(\Delta)$ meets the open unit disk Δ.*

Proof. Suppose $f(\Delta)$ contains Δ. Then $g = f^{-1}$ is defined on Δ, $g(\Delta) \subset \Delta$, $g(0) = 0$, and $g'(0) = 1/f'(0) = 1$. By the Schwarz lemma, $g(\zeta) \equiv \zeta$, and so $f(z) \equiv z$. □

page 57, Theorem III.1.8

Another way to organize the proof that the Julia set $\mathcal{J}$ has no isolated points is as follows. Theorem 1.5 shows that if U is an open set that meets $\mathcal{J}$, then $\mathcal{J} \subset \cup_{n=0}^{\infty} R^n(U)$. From compactness we then obtain the following.

Lemma *If U is an open set that meets $\mathcal{J}$, then there is $N \ge 0$ such that*

$$\mathcal{J} \ \subset\ U \cup R(U) \cup \cdots \cup R^N(U).$$

Now to prove Theorem 1.8, we assume that z_0 is an isolated point of $\mathcal{J}$, and we take U to be an open set such that $U \cap \mathcal{J} = \{z_0\}$. By the lemma, there is N such that $\mathcal{J} = \{z_0, R(z_0), \dots, R^N(z_0)\}$. If $m = (N+1)!$, then each $R^k(z_0)$ is a fixed point for R^m. Consequently $\{z_0\}$ is completely invariant for R^m. This places z_0 in the Fatou set, which is a contradiction.

page 71, Theorem IV.1.3 (Sullivan's Theorem)

The argument in the first paragraph of the proof, based on $\sum \mathrm{area}\, U_n < \infty$, can be simplified. Since each $R^n(z)$ lies in a different component of the Fatou set, the only limit points of $\{R^n(z)\}$ are in the Julia set. Consequently any limit function of $\{R^n\}$ on U_0 takes its values in the Julia set and hence is constant.

page 124, Theorem VIII.1.2

A fast proof that the Mandelbrot set $\mathcal{M}$ is contained in the disk $\{|c| \le 2\}$ proceeds as follows. For $c \in \mathcal{M}$, set $B = \sup_{n\ge1} |P_c^n(0)|$. Note that $|c| = |P_c(0)| \le B$. From $P_c^n(0)^2 = P_c^{n+1}(0) - c$ we obtain $|P_c^n(0)|^2 \le B + |c| \le 2B$. Thus $B^2 \le 2B$, and $B \le 2$.