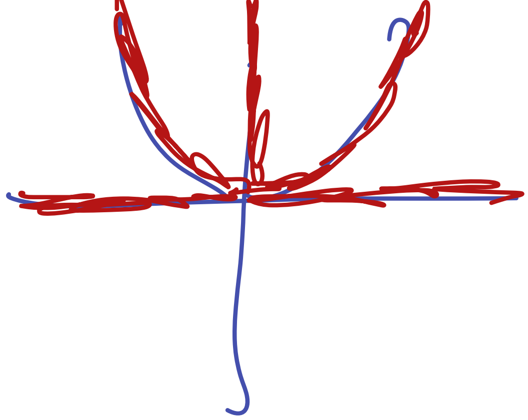


$H(x,y)$ s.t. $x' = f(x,y)$, $y' = g(x,y)$

$$\frac{\partial H}{\partial y} = f(x,y)$$

$$\frac{\partial H}{\partial x} = -g(x,y)$$

$$\frac{d}{dt} H(x(t), y(t)) = 0$$

Phase Portraits for non-linear Systems:

$$x' = f(x,y)$$
$$y' = g(x,y)$$

- local method: look at linearization near eq. pts, usually says what system looks like near the pt.

- Global method: use info from nullclines to get info about solution trajectories

Generally want to combine both techniques to get an accurate phase portrait sketch

$$x' = y - x^3$$
$$y' = x - y$$

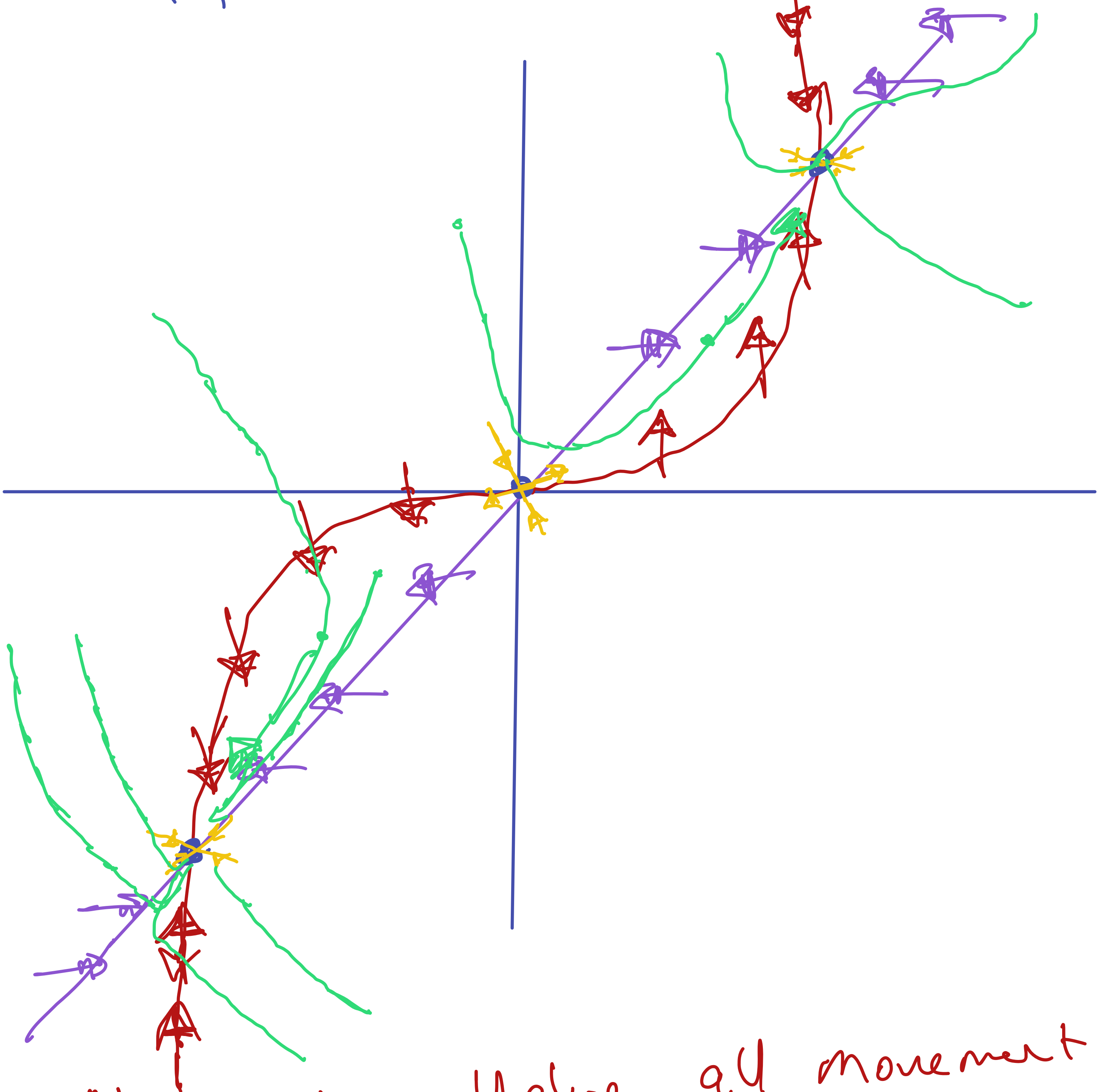
What does a phase portrait look like?

First, find EQ pts:

$$y - x^3 = 0 \Rightarrow y = x^3$$
$$x - y = 0 \Rightarrow x = x^3$$

$$\Rightarrow x = \pm 1, 0 \quad , \quad y = \pm 1, 0$$

EQ pts: $(-1,-1), (0,0), (1,1)$.



Along x-nullcline, all movement is vertical.

$y = x^3 \Rightarrow$ $\frac{dy}{dt} = x - x^3$

Along y-nullcline, $y = x$

$$\Rightarrow \quad x' = x - x^3$$

Now, let's look locally at EQ points.

$$J = \begin{pmatrix} -3x^2 & 1 \\ 1 & -1 \end{pmatrix}$$

$(-1,1)$: $\quad J = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$

$T = -4 \qquad D = 2$

Nodal Sink

$(0,0)$: $\quad J = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$T = -1 \quad D = -1$

Saddle

$(1,1)$: $\quad J = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$

Nodal Sink

When EQ points are not generic, it's generally very hard to get an accurate phase portrait.

There are some exceptions:

- $$x' = f(x,y)$$
  $$y' = g(x,y)$$

and f, g have cts second partials, then degenerate nodes stay degenerate nodes.

- If you can find

H w/

$$\frac{dH}{dy} = f(x,y)$$

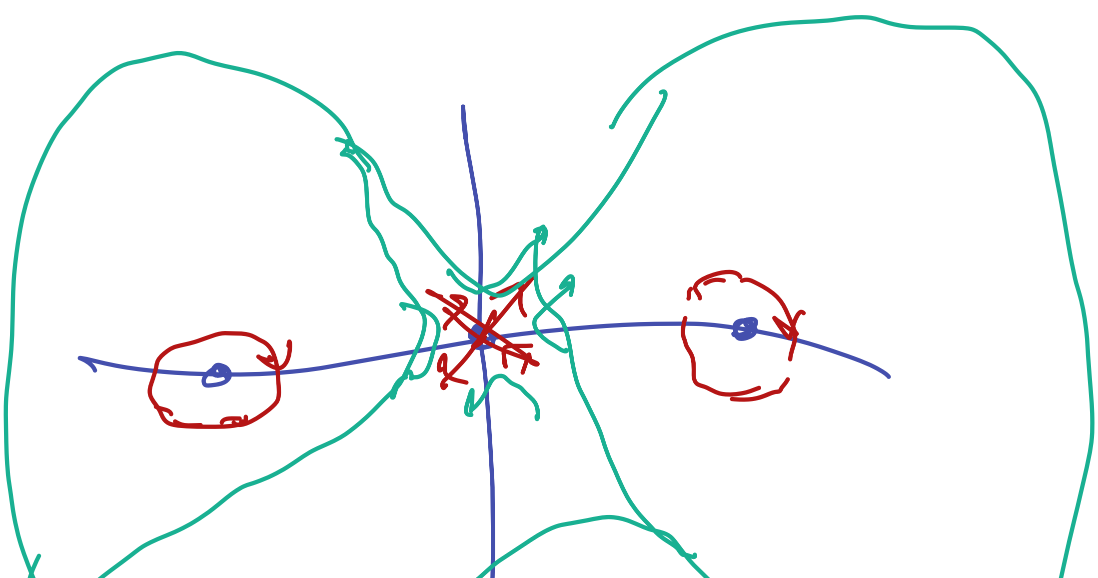
$$-\frac{dH}{dx} = g(x,y)$$

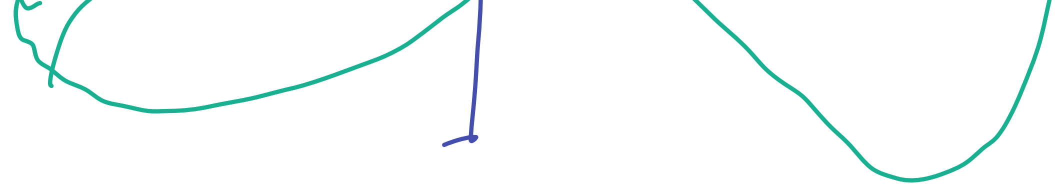
then a center remains a center.

E.g. #2 on HW 4.

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = y - y^3 .$$





How to get phase portraits of saddles:

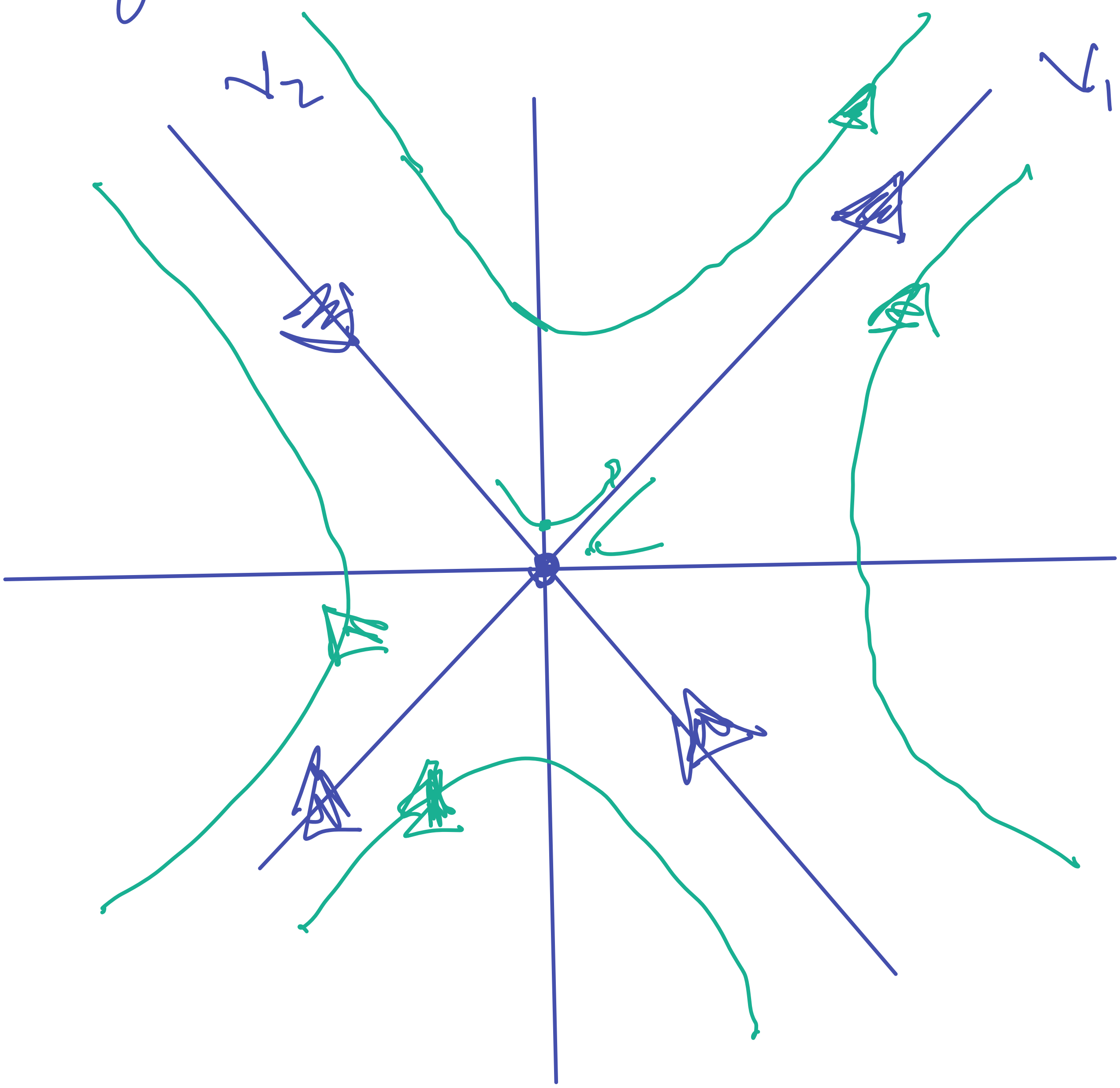
$Y' = AY$ Saddle

when eigenvalues have opposite signs. Equivalently, $\det(A) < 0$.

$\lambda_1 > 0$ v_1

$\lambda_2 < 0$ v_2

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$$

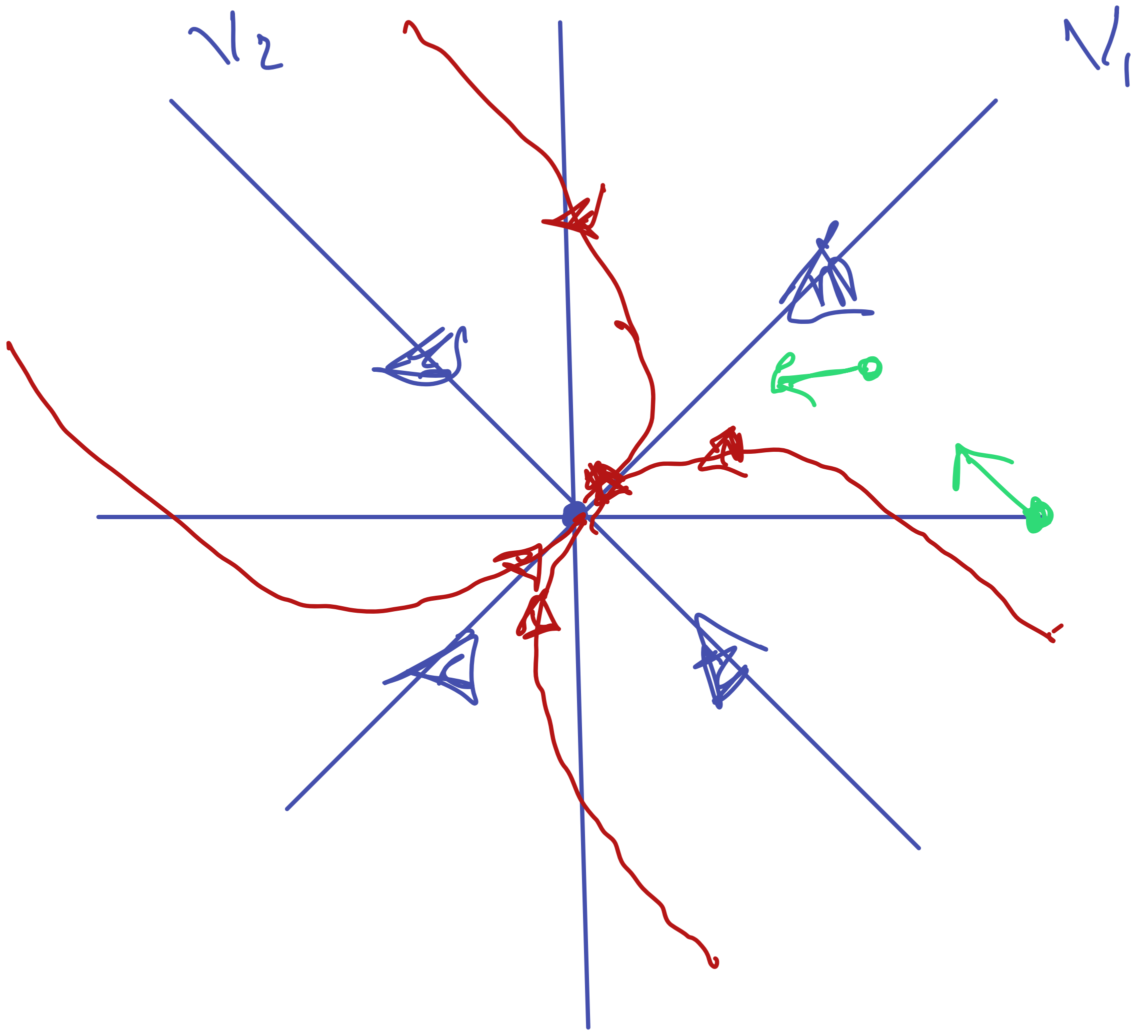


As $t \to \infty$,

$$y(t) \approx c_1 e^{\lambda_1 t} v_1$$

As $t \to -\infty$,

$y(t) \approx c_2 e^{\lambda_2 t} v_2$



Nodal Sink: $\lambda_1, \lambda_2 < 0$

$0 > \lambda_1 > \lambda_2$

$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

As $t \to \infty$,

$y(t) \approx c_1 e^{\lambda_1 t} v_1$

b/c second term goes

to 0 faster.

Picking test points is

often very useful

for figuring out

what trajectories

are doing!

e.g. for determining rotation direction for centers / spirals.

Degenerate Nodes:

λ repeated

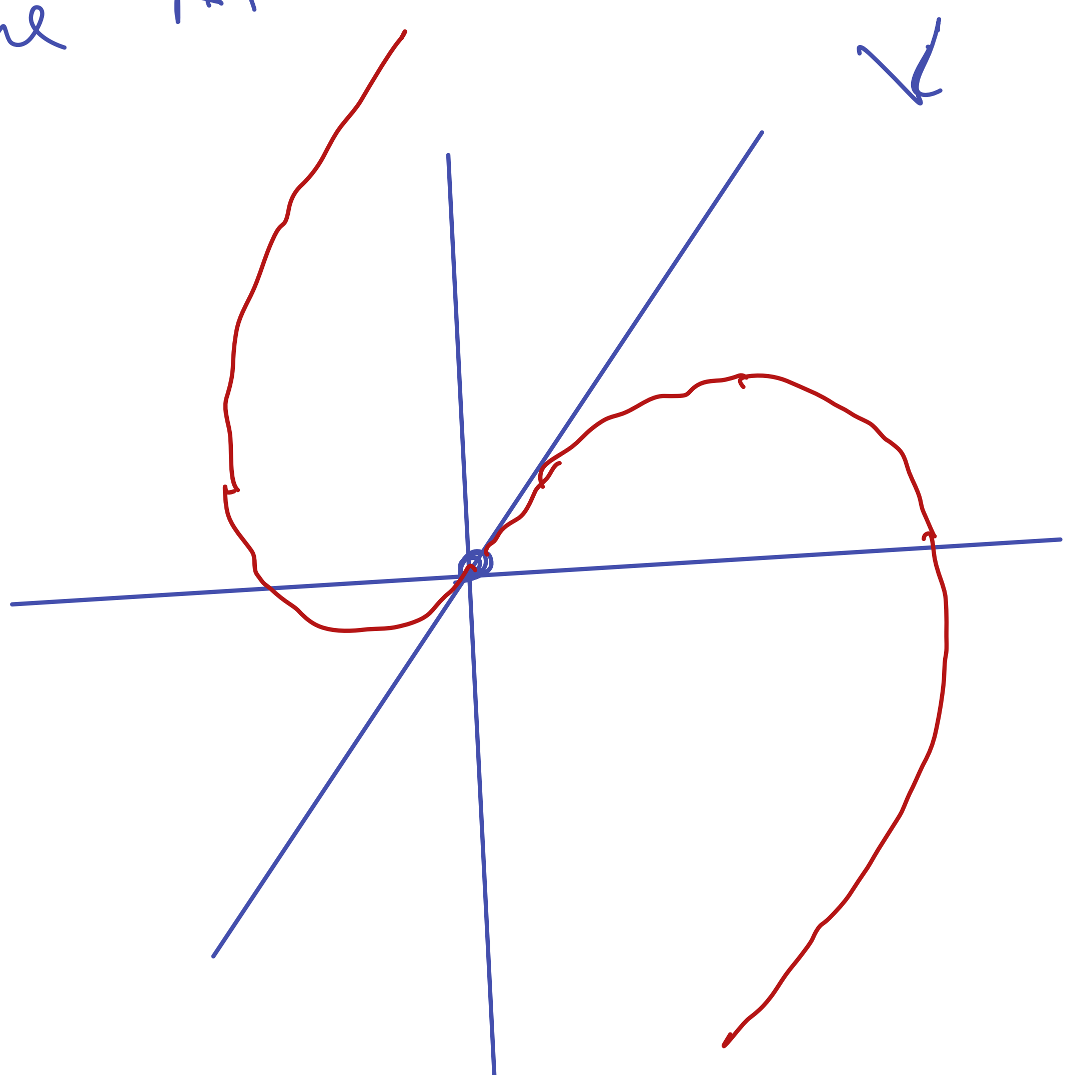
Source or Sink depending on sign of λ. ✓

$$y(t) = c_1 e^{\lambda t} v + c_2 e^{\lambda t}(v_2 + tv)$$

As $t \to \infty$

$$y(t) \approx c_2 e^{\lambda t} t v$$

Same for $t \to -\infty$



Again can pick test points to see what vector field says you should do.

$$T^2 = 4D$$

Complex eigenvalues:

$$\lambda = a + bi$$

$$(A - \lambda I) v = 0$$

$$Y' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} Y$$

$$\lambda = -1 + 2i$$

$$(A - \lambda I) v = 0$$

$$\begin{pmatrix} 2-2i & -4 \\ 2 & -2-2i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-2i)x - 4y = 0$$

$$2x - (2+2i)y = 0$$

$(2-2i)x = 4y$

$$\boxed{y = \left(\tfrac{1}{2} - \tfrac{1}{2}i\right)\cdot x}$$

$x = 2$ $\quad$ $y = 1-i$

$v = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$ is an eigenvector.

What's general sol^n?

$$y(t) = c_1 e^{(-1+2i)t}\begin{pmatrix} 2 \\ 1-i \end{pmatrix} + c_2 e^{(-1-2i)t}\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$$

How to write in terms of

sin/cos:

$$e^{-t}\left((\cos(2t) + i\sin(2t))\left(\begin{pmatrix}2\\1\end{pmatrix} + i\begin{pmatrix}0\\-1\end{pmatrix}\right)\right)$$

taking real/imaginary parts

give real fundamental sol'ns.

$$e^{-t}\cos(2t)\begin{pmatrix}2\\1\end{pmatrix} - e^{-t}\sin(2t)\begin{pmatrix}0\\1\end{pmatrix}$$

$$e^{-t}\cos(2t)\begin{pmatrix}0\\-1\end{pmatrix} + e^{-t}\sin(2t)\begin{pmatrix}2\\1\end{pmatrix}$$

Variation of Parameter for

Systems:

$$Y' = AY + F$$

- Find Fundamental matrix

$$Y_f = \begin{bmatrix} | & & | \\ Y_1 & \cdots & Y_n \\ | & & | \end{bmatrix}$$

Y_i are fundamental sol^n to $Y' = AY$.

- Hunt for a solution of the form $Y_f \cdot V$ for some V.

$$V' = Y_f^{-1} F$$

$$V = \int Y_f^{-1} F \, dt$$

Integral is component wise

$$\boxed{Y_f \cdot \int Y_f^{-1} F \, dt = Y_p}$$

$$Y' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} Y + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

$$Y' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} Y$$

$$y_1 = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$y_2 = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$Y_f = \begin{pmatrix} -e^{-t} & e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix}$$

$$Y_f^{-1} = \begin{pmatrix} -e^{t}/2 & e^{t}/2 \\ e^{-3t}/2 & e^{-3t}/2 \end{pmatrix}$$

$$Y_f^{-1} F = \begin{pmatrix} \cdots \end{pmatrix} \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ e^{-4t}/2 \end{pmatrix} = V'$$

$$\Rightarrow \quad V = \begin{pmatrix} -t/2 \\ -e^{-4t}/8 \end{pmatrix}$$

$$Y_p = Y_f V = \begin{pmatrix} \cdots \end{pmatrix}$$

$$= \frac{1}{8} \begin{pmatrix} (-1+4t)e^{-t} + e^{3t} \\ (-1-4t)e^{-t} + e^{3t} \end{pmatrix}$$

Key property of Y_f that makes variation of parameters work:

$$Y_f' = AY_f. \quad \text{Why?}$$

$$Y_f = \begin{bmatrix} | & & | \\ Y_1 & \cdots & Y_n \\ | & & | \end{bmatrix}$$

$$AY_f = \begin{bmatrix} | & & | \\ AY_1 & \cdots & AY_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ Y_1' & \cdots & Y_n' \\ | & & | \end{bmatrix} = Y_f'.$$

Competing Species / Predator-Prey

Examples of nonlinear

Systems.

Lotka-Volterra

Predator-Prey

$$F' = (a - bS)F$$

$$S' = (-c + dF)S$$

$$R' = \overbrace{K_1 R\left(1 - R/L_1\right)}^{\text{logistic growth}} - C_1 RS$$

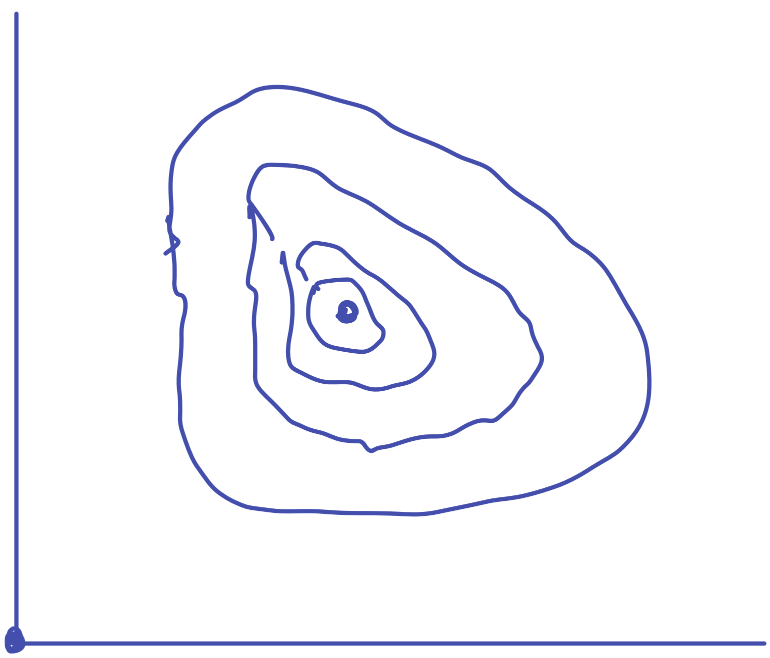
$$S' = \underbrace{K_2 S\left(1 - S/L_2\right)}_{\text{logistically}} - C_2 RS$$

Competing Species

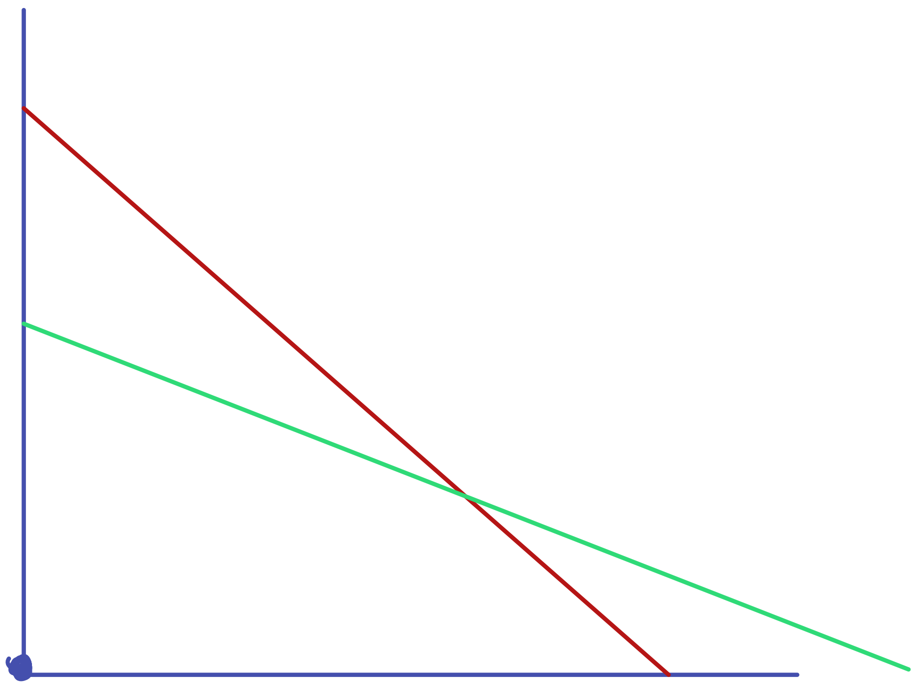
Key difference: 1 is a predator

prey model, one species dies in the absence of the other. For competing species, both pop. can thrive if the other doesn't exist.

Lokta - Volterra



Competing Species



Typically what ends up happening is that either the species can coexist, or one species dies off from being out competed