Higher order ODEs

Matrix exponential tells us how to solve linear systems

$Y' = AY$ A $n \times n$ matrix.

$$y^{(n)} + a_1(t)\, y^{(n-1)} + \dots + a_{n-1}(t)\, y = 0$$

homogeneous n^{th} order linear ODE.

What techniques can we use to solve these / how does the theory work?

Any ODE can turn into a linear system as follows:

$$x_1 = y$$
$$x_2 = y'$$
$$\vdots$$
$$x_n = y^{(n-1)}$$

$\rightsquigarrow$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_n' = -a_1 x_n - \cdots - a_n x_1 \end{cases}$$

Ex for a 3rd order ODE:

$$y''' + a_1 y'' + a_2 y' + a_3 y = 0$$

$$\begin{cases} y' = v \\ v' = w \\ w' = -a_1 w - a_2 v - a_3 y \end{cases}$$

As a result of this, all the theory of systems

Carries over to give us the things we want.

- Existence and Uniqueness thm
- linear combinations of sol^{n}s are still sol^{n}s
- Degree n ODE has n fundamental solutions that make up the general solution.

Can always use the matrix exponential techniques from yesterday to solve the system.

However, this is cumbersome

esp. if the degree of ODE is big. We'll talk about how to use the general theory to avoid doing this.

We'll assume that

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0$$

correspond to the system

$$Y' = AY.$$ Sol^{n}s to this system are vectors that look like

$$\begin{pmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix}$$

So we really only care about

first component.

The Wronskian: if $y_1, \dots, y_n$ are solutions to the ODE, we define the Wronskian of them by

$$W = \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \ddots & y_n' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Like before, $y_1, \dots, y_n$ are linearly independent sol^n $\iff$ $W \neq 0$.

In system land, $Y_i = \begin{pmatrix} y_i \\ y_i' \\ \vdots \\ y_i^{(n-1)} \end{pmatrix}$

Checking if $y_1, \dots, y_n$ are l.i. is equivalent to checking if

$Y_1, \ldots, Y_n$ are l.i. sol'n to $Y' = AY$.

From linear algebra, this happens iff $\det \begin{bmatrix} | & & | \\ Y_1 & \cdots & Y_n \\ | & & | \end{bmatrix} \neq 0$.

this is just the Wronskian!

Constant Coeff. ODEs

$$y^{(n)} + a_1 y^{(n-1)} \cdots + a_n y = 0$$

a_i are constants.

Like before, don't have general techniques to solve non-constant coeff. equations very easily.

Recall that sol^n to the corresponding system $Y' = AY$ can be found by finding solutions coming from generalized eigenvectors of A.

$$Y_i(t) = e^{\lambda t} e^{(A-\lambda I)t} v_i$$

v_i generalized eigenvectors for λ of degree p

$$Y_i(t) = e^{\lambda t}\left[I + (A-\lambda I)t + \dots + \frac{1}{(p-1)!} t^{p-1} (A-\lambda I)^{p-1}\right] v_i$$

Constructing enough of these sol^{n}s lets us get the general sol^n.

We can define the characteristic polynomial of the ODE by

$$\lambda^n + a_1\lambda^{n-1} + \dots + a_n$$

this turns out to be the same as char poly. of the system!

let's say that v is an eigenvector for λ. Then we get a sol^n that looks like $Y(t) = e^{\lambda t} v$

So we get a corresponding sol^n

$y(t) = e^{\lambda t}$ to the ODE.

When λ is a repeated root, the

Sol'ns coming from generalized eigenvectors in first component look like $e^{\lambda t} p(t)$ for some $p(t)$ of degree $\leq p-1$.

So we are led to believe that if λ is a repeated root of mult. p, that we get sol'ns that look like:

$$e^{\lambda t}, \; te^{\lambda t}, \; t^2 e^{\lambda t}, \ldots, \; t^{p-1} e^{\lambda t}$$

Can use linear algebra to show these are indeed solutions, so that a general solution looks like a linear combination of them.

Ex: $y^{(4)} - 2y''' + 2y' - y = 0$

Characteristic polynomial:

$$\lambda^4 - 2\lambda^3 + 2\lambda - 1 =$$

$$(\lambda - 1)^3 (\lambda + 1).$$

roots: -1, $+1$ of mult 3.

get fundamental sol^{n}s:

$y_1(t) = e^{-t}$

$y_2(t) = e^{t}$

$y_3(t) = te^{t}$

$y_4(t) = t^2 e^{t}$

general sol^n:

$c_1 e^{-t} + c_2 e^{t} + c_3 t e^{t} + c_4 t^2 e^{t}$

Complex roots have the same story. If λ complex root of char. poly. of mult. q then

$e^{\lambda t}, te^{\lambda t}, \ldots, t^{q-1} e^{\lambda t}$ and

$e^{\bar{\lambda} t}, te^{\bar{\lambda} t}, \ldots, t^{q-1} e^{\bar{\lambda} t}$ are

all sol^n to the ODE

taking real/imaginary parts,

we produce the solutions

$$e^{\alpha t}\cos(\beta t),\ t e^{\alpha t}\cos(\beta t), \dots, t^{q-1} e^{\alpha t}\cos(\beta t)$$

$$e^{\alpha t}\sin(\beta t), \dots, t^{q-1} e^{\alpha t}\sin(\beta t($$

$$\lambda = \alpha + i\beta$$

Ex: $y^{(4)} + 4y''' + 14y'' + 20y' + 25y = 0$

Char poly:

$$\lambda^4 + 4\lambda^3 + 14\lambda^2 + 20\lambda + 25$$

$$(\lambda^2 + 2\lambda + 5)^2$$

roots: $-1 \pm 2i$ mult. 2

$y_1(t) = e^{-t}\cos(2t)$

$y_2(t) = e^{-t}\sin(2t)$

$y_3(t) = t\,e^{-t}\cos(2t)$

$y_4(t) = t\,e^{-t}\sin(2t)$

general sol^n:

$c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$

10 min break

Inhomogeneous Systems

$Y' = AY + F$

$A = A(t)$
$F = F(t)$

How do we solve such systems?

Very similar to what we did before.

Thm: Y_h general sol^n to $Y' = AY$ and Y_p is a particular sol^n, then the general sol^n to $Y' = AY + F$ is

$Y_h + Y_p$.

Proof is same as before.

<u>Def.</u> $Y_1, \dots, Y_n$ a fundamental sol^n set to $Y' = AY$ then the <u>fundamental matrix</u> is

$$Y_f = \begin{bmatrix} | & & | \\ Y_1 & \cdots & Y_n \\ | & & | \end{bmatrix}$$

i.e. the matrix whose columns are the vectors $Y_1, \dots, Y_n$.

<u>Variation of Parameter</u>

Look for Y_p of the form

$Y_f(t)v(t)$ for some

function $v(t)$.

$$\begin{pmatrix} | & & | \\ Y_1 & \cdots & Y_n \\ | & & | \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}$$

$$Y_p(t) = Y_1(t)V_1(t) + \cdots + Y_n(t)V_n(t).$$

So we're hunting for functions

$V_1(t), \ldots, V_n(t)$ so that Y_p

is a non-constant linear

Combination of fundamental

Solutions.

What we need:

$Y_p' = AY_p + F.$ Let's see

what that means here,

$Y_p = Y_f V$ So

$$Y_p' = (Y_f V)' = Y_f V' + Y_f' V$$

$\Rightarrow$

$$Y_f V' + Y_f' V = AY_f V + F$$

Note: $AY_f = Y_f'$. Why?

$$Y_f = \begin{bmatrix} | & & | \\ Y_1 & \cdots & Y_n \\ | & & | \end{bmatrix}$$

$$AY_f = \begin{bmatrix} | & & | \\ AY_1 & \cdots & AY_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ Y_1' & \cdots & Y_n' \\ | & & | \end{bmatrix}$$

$$= Y_f' .$$

$$\Rightarrow Y_f V' + Y_f' V = Y_f' V + F$$

$$\Rightarrow Y_f V' = F$$

$$\Rightarrow V' = Y_f^{-1} F$$

$$\Rightarrow V = \int Y_f^{-1} F \, dt$$

$$\boxed{Y_p = Y_f \cdot V = Y_f \cdot \int Y_f^{-1} F \, dt}$$

<u>Example</u> Find a particular sol^n to $Y' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} Y + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$

First, solve $Y' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} Y$ to get fundamental matrix.

$$p_A(x) = x^2 - 2x - 3$$
$$= (x+1)(x-3)$$

$\lambda = -1, 3$ eigenvectors

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Fundamental Sol'^s:

$$Y_1(t) = e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}$$

$$Y_2(t) = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$$

Fundamental matrix:

$$\begin{pmatrix} -e^{-t} & e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix} = Y_f$$

$$Y_f^{-1} = -\frac{1}{2e^{2t}}\begin{pmatrix} e^{3t} & -e^{3t} \\ -e^{-t} & -e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{t}/2 & e^{t}/2 \\ e^{-3t}/2 & e^{-3t}/2 \end{pmatrix}$$

$$Y_f^{-1} \cdot F = \begin{pmatrix} -e^{t}/2 & e^{t}/2 \\ e^{-3t}/2 & e^{-3t}/2 \end{pmatrix} \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ e^{-4t}/2 \end{pmatrix} = v'$$

$$\Rightarrow v = \int \begin{pmatrix} -1/2 \\ e^{-4t}/2 \end{pmatrix} dt$$

$$v = \begin{pmatrix} -t/2 \\ -e^{-4t}/8 \end{pmatrix}$$

$$Y_p = Y_f \cdot v$$

$$= \begin{pmatrix} -e^{-t} & e^{3t} \\ e^{-t} & e^{3t} \end{pmatrix} \begin{pmatrix} -t/2 \\ -e^{-4t}/8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{t}{2} e^{-t} - e^{-t}/8 \\ -\frac{t}{2} e^{-t} - e^{-t}/8 \end{pmatrix}.$$

Some Remarks.

How can you solve a hom.
System where coeff. matrix

is not constant?

$$Y' = A(t)\,Y$$

Can still use matrix exponential!

If $B(t) = \int A(t)\,dt$

then

$$\frac{d}{dt} e^{B(t)} = B'(t)\,e^{B(t)}$$

$$= A(t)\,e^{B(t)}$$

So $e^{B(t)}$ is a sol^n to

$Y' = A(t)Y$. If $A(t)$ is diagonalizable, can use the method from yesterday to easily compute $e^{B(t)}$.

<u>Ex:</u> $Y' = A(t)Y$

$$A(t) = \begin{pmatrix} 1 & -\cos(t) \\ \cos(t) & 1 \end{pmatrix}$$

$$B(t) = \begin{pmatrix} t & -\sin(t) \\ \sin(t) & t \end{pmatrix}$$

$$P_B(x) = x^2 - 2tx + (t^2 + \sin^2(t))$$

Eigenvalues:

$$\lambda = t \pm i\sin(t)$$

Eigenvectors:

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$B = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t+i\sin(t) & 0 \\ 0 & t-i\sin(t) \end{pmatrix} \begin{pmatrix} -1/2 i & 1/2 \\ 1/2 i & 1/2 \end{pmatrix}$$

$$e^{B(t)} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{t+i\sin(t)} & 0 \\ 0 & e^{t-i\sin(t)} \end{pmatrix} \begin{pmatrix} -1/2 i & 1/2 \\ 1/2 i & 1/2 \end{pmatrix}$$

If you do all the

relevant simplifications,

you get

$$e^{B(t)} = e^{t}\begin{pmatrix} \cos(\sin(t)) & -\sin(\sin(t)) \\ \sin(\sin(t)) & \cos(\sin(t)) \end{pmatrix}$$

This technique is actually a general way of solving

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = 0!$$

Convert to system, use matrix exponential, recover first component of sol^{n}

to get $y(t)$.

Remark 2: B/c variation of parameter works for system, works for any ODE.

Let's see how it works for 3^{rd} order ODE.

$$y''' + a_1 y'' + a_2 y' + a_3 y = f(t)$$

$$\begin{cases} y' = v \\ v' = w \end{cases}$$

$$w' = -a_1 w - a_2 v - a_3 y$$

$$Y' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} \begin{pmatrix} y \\ v \\ w \end{pmatrix} + \begin{pmatrix} f(t) \\ 0 \\ 0 \end{pmatrix}$$

Suppose y_1, y_2, y_3 are fundamental to homogeneous ODE.

$$Y_f = \begin{pmatrix} | & | & | \\ Y_1 & Y_2 & Y_3 \\ | & | & | \end{pmatrix}$$

$$Y_i = \begin{pmatrix} y_i \\ y_i' \\ y_i'' \end{pmatrix}$$

$$V' = Y_f^{-1} \cdot F = Y_f^{-1} \cdot \begin{pmatrix} f(t) \\ 0 \\ 0 \end{pmatrix}$$

$$Y_f^{-1} = \frac{1}{\det(Y_f)} \operatorname{Adj}(Y_f)$$

$$= \frac{1}{W} \operatorname{Adj}(Y_f)$$

$$\operatorname{Adj}(Y_f) = \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix}$$

$$v' = \frac{1}{\omega} \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} f(t) \\ 0 \\ 0 \end{pmatrix}$$

$$v' = \frac{1}{\omega} \begin{pmatrix} f(t)\, c_{11} \\ f(t)\, c_{12} \\ f(t)\, c_{13} \end{pmatrix}$$

$$Y_f = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}$$

Note that e.g. $C_{11} = \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix}$

$W_i = C_{1i}$, so

$$V' = \frac{1}{W} \begin{pmatrix} W_1 f(t) \\ W_2 f(t) \\ W_3 f(t) \end{pmatrix}$$

$$\Rightarrow V = \int \begin{pmatrix} \frac{W_1}{W} f(t) \\ \frac{W_2}{W} f(t) \\ \frac{W_3}{W} f(t) \end{pmatrix} dt$$

$$Y_p = Y_f \cdot V$$

$$= y_1 \int \frac{\omega_1}{\omega} f(t)\, dt + y_2 \int \frac{\omega_2}{\omega} f(t)\, dt + y_3 \int \frac{\omega_3}{\omega} dt$$