The Matrix Exponential

Set up: $Y' = AY$ $\quad A$ $n \times n$ matrix.

1D: $y' = ay \implies y(t) = Ce^{at}$

2D: Y generally looks like linear combinations of exponential solutions.

<u>Def:</u> For an $n \times n$ matrix A, we define the matrix exponential e^A by

$$e^A = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

One can show that this series converges to a matrix, so that e^A is actually well-defined.

With this definition, it's easy to prove the higher dimensional analogue of the 1D case.

Thm:

- $\frac{d}{dt} e^{At} = A e^{At}$
- The sol^n to $Y' = AY$ w/ $Y(0) = V$ is given by $Y(t) = e^{At} V$.

Proof:

- $$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{k=0} \frac{1}{k!} (At)^k$$

$$= \frac{d}{dt} \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} A^k t^{k-1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} A^{k+1} t^k = A \sum_{k=0}^{\infty} \frac{1}{k!} A^k t^k$$

$$= A e^{At}.$$

- Since $Y(t) = e^{At} v$ solves the IVP $Y'(t) = AY(t)$ $Y(0) = v$, the uniqueness thm says this is <u>the</u> sol^n.

Theoretically, this says if we can compute e^{At} we can solve any system. Let's start with some easy cases.

1. $A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ is diagonal.

$$A^k = \begin{pmatrix} a_1^k & & \\ & \ddots & \\ & & a_n^k \end{pmatrix}$$

$$e^A = I + A + \frac{1}{2}A^2 + \cdots$$

$$= \begin{pmatrix} 1 + a_1 + \frac{1}{2}a_1^2 + \cdots & & \\ & \ddots & \\ & & 1 + a_n + \frac{1}{2}a_n^2 + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{a_1} & & \\ & \ddots & \\ & & e^{a_n} \end{pmatrix}$$

So matrix exponential is very easy for diagonal matrix.

2. Next, we suppose that A is diagonalizable. This means that A has a basis of eigenvectors, so w.r.t. this basis, A is diagonal, via change of basis, this means

$$A = P D P^{-1}$$

D = diagonal matrix w/ eigenvalues

on diagonal

P = matrix whose columns are eigenvectors for each eigenvalue.

We're not going to focus on A larger than 3×3 b/c computationally gets hard.

When $A = PDP^{-1}$, we can compute e^A as follows:

$$e^A = e^{PDP^{-1}} = \sum_{k=0}^{\infty} \frac{1}{k!}(PDP^{-1})^k$$

$$(PDP^{-1})^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

in general,

$$(PDP^{-1})^k = PD^kP^{-1}$$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} PD^kP^{-1} = P\left(\sum_{k=0}^{\infty} \frac{1}{k!} D^k\right)P^{-1}$$

$$= \boxed{Pe^{P}P^{-1}}$$

$$Y(t) = e^{At}v = e^{PDtP^{-1}}v$$

$$= Pe^{Dt}P^{-1}v$$

Quick review: how to compute inverse of matrix.

- $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad P^{-1} = \dfrac{1}{\det(P)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

- P 3×3 $\quad P^{-1} = \dfrac{1}{\det|P|} \mathrm{Adj}(P)$

$$\mathrm{Adj}(P) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T$$

$C_{ij} = (i,j)^{th}$ cofactor

$= (-1)^{i+j} \cdot \det\left((i,j)^{th} \text{ minor}\right)$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

eigenvalues: $\lambda = 1, 3$

eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$e^{At} = Pe^{Dt}P^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t} & \\ & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{t} & -e^{t} \\ 0 & e^{3t} \end{pmatrix} = \begin{pmatrix} e^{t} & -e^{t} + e^{3t} \\ 0 & e^{3t} \end{pmatrix}$$

A sol^n to $Y' = AY$ w/ $Y(0) = v$
$= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$Y(t) = \begin{pmatrix} e^{t} & -e^{t} + e^{3t} \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{t} + c_2(-e^{t} + e^{3t}) \\ c_2 e^{3t} \end{pmatrix}$$

$$= (c_1 - c_2) e^{t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$p_A(x) = (x-2)(x-3)^2$$

Eigenvalues: $\lambda = 2, 3$

$\lambda = 2$: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda = 3$: $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$$(A - 3I) = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x = 3z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3z \\ y \\ z \end{pmatrix}$$

$$= y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & & \\ & 3 & \\ & & 3 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

to compute P^{-1}:

$\det(P) = 1$

$$\mathrm{Adj}(P) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}^T$$

to solve $Y' = AY \quad Y(0) = V$

$$e^{At} =$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & & \\ & e^{3t} & \\ & & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & -3e^{2t} \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} & 0 & -3e^{2t}+3e^{3t} \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

$$Y(t) = e^{At} v.$$

10 min Break

What do you do when A is not diagonalizable?

Can use something called "Jordan Canonical Form" which is a different factorization. Details end up similar to above.

For us, we'll try something else (although it ends up being very similar).

Let's examine e^A a little more carefully.

- If $A^k v = 0$ for some k and v,

then $e^{At} v$

$$= \left(I + At + \frac{1}{2} A^2 t^2 + \dots + \frac{1}{(k-1)!} A^{k-1} t^{k-1} \right) v$$

- e^{A} satisfies $e^{A+B} = e^{A} \cdot e^{B}$

when A and B commute,

i.e. $AB = BA$.

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{1}{k!} (A+B)^k$$

$$(A+B)^2 = A^2 + \underbrace{AB + BA} + B^2$$

$$= A^2 + 2AB + B^2.$$

etc.

then b/c this works at the level of power series for the real #'s by just doing the mult. it works for matrices too b/c have the necessary algebraic properties.

Ultimately, we want to solve $Y' = AY$, $Y(0) = v$.

By general theory, if we

Can find n-linearly independent sol^n to $y' = Ay$, then an arbitrary sol^n has to be a linear combination of them.

<u>Def</u>: A <u>generalized eigenvector</u> of A is a vector v s.t.

$$(A - \lambda I)^p v = 0 \quad \text{for some } p \geq 1.$$

this is going to motivate the following approach.

For any λ, we can write

$$tA = t\lambda I + (A-\lambda I)t$$

$$e^{At} = e^{\lambda I t + (A-\lambda I)t}$$

$$= e^{\lambda I t} \cdot e^{(A-\lambda I)t} = e^{\lambda t} \cdot e^{(A-\lambda I)t}$$

If v is a generalized eigenvector w/ exponent p,

$$e^{(A-\lambda I)t} v = \left(I + t(A-\lambda I) + \frac{1}{2}t^2(A-\lambda I)^2 + \cdots + \frac{1}{(p-1)!}t^{p-1}(A-\lambda I)^{p-1} \right) v$$

So $e^{At}v$ is then a solution that we can compute using the above.

Ex: $A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 0 \\ -1 & -3 & -3 \end{pmatrix}$

and we want to solve $Y' = AY$.

$p_A(x) = (x+1)(x+2)^2$

Eigenvalues: $\lambda = -1, -2$

$\lambda = -1$: $\quad v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\lambda = -2$: $\quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Not diagonalizable, not enough l.i. eigenvectors.

Since v_1, v_2 are eigenvectors, we get corresponding solutions

$$y_1(t) = e^{-t} v_1 = e^{-t}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$y_2(t) = e^{-2t} v_2 = e^{-2t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

To find a third solution $y_3(t)$, we need a generalized eigenvector.

$$A+2E = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix}$$

$$(A+2I)^2 = \begin{pmatrix} 0 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & -8 & 0 \end{pmatrix}$$

a vector in the kernel that's l.i. from the eigenvectors that we have found is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = v_3$

A solution is

$$y_3(t) = e^{-2t} \cdot e^{(A+2I)t} v_3$$

$$e^{(A+2I)t} v_3 = \left[I + t \cdot (A+2I) \right] v_3$$

$$y_3(t) = e^{-2t} \left[v_3 + t(A+2I)v_3 \right]$$

$$\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right]$$

$$= e^{-2t} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right]$$

Note that solutions

$y_1(t)$, $y_2(t)$, and $y_3(t)$ are all linearly independent, b/c at $t=0$ the vector

$\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ are

linearly independent.

$$\Rightarrow \quad Y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t).$$

The Algorithm:

$Y' = AY$ $\quad A$ $n \times n$ matrix

- A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ w/ alg. multiplicities $q_1, \dots, q_k$. Note that $q_1 + \dots + q_k = n$.

For each eigenvalue λ:

- Find the smallest p s.t. $(A - \lambda I)^p$ has a q dimensional kernel. Find a basis $\{v_1, \dots, v_q\}$.

Set $y_j(t) = e^{At} v_j$

$$= e^{\lambda t}\left(v_j + (A-\lambda I)t v_j + \cdots + \frac{1}{(p-1)!} t^{p-1} (A-\lambda I)^{p-1} v_j \right)$$

All these sol^n are linearly ind.

$$\Rightarrow \quad Y(t) = \sum_{i=1}^{n} c_i Y_i(t)$$

<u>Ex:</u> $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Want to solve $Y' = AY$.

$$p_A(x) = (x-1)^3$$

$\lambda = 1$ eigenvalue

$A - I$: $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ dim ker = 1

$(A-I)^2$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ dim ker = 2

$(A-I)^3$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ dim ker = 3

Since $(A-I)^3 = 0 \Rightarrow$

basis for kernel is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$y_1(t) = e^t \cdot e^{(A-I)t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$ b/c $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is

eigenvectors.

$$y_2(t) = e^t \cdot e^{(A-I)t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= e^t \left(I + (A-I)t \right) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$y_3(t) = e^t \cdot e^{(A-I)t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= e^t \left(I + (A-I)t + \frac{1}{2}t^2 (A-I)^2 \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$Y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t).$$

general sol^n.