Math 33B 7/26

Matrix Exponential

$$y' = ay \qquad y = e^{at} \cdot C$$

$$y' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} y$$

$$y' = Ay \qquad y = e^{At} v$$

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$$\frac{d}{dt} e^{tA} = A e^{tA}$$

$e^{tA}v$

λ $Av = \lambda v$ $(A - \lambda I)v = 0$

$$e^{tA}v = Iv + tAv + \frac{t^2}{2!}A^2v + \dots + \frac{t^n}{n!}A^nv + \dots$$

$$= v + t\lambda v + + \frac{t^2}{2!}\lambda^2 v + \dots + \frac{t^n}{n!}\lambda^n v + \dots$$

$$= \left(1 + t\lambda + \frac{(\lambda t)^2}{2!} + \dots + \frac{(\lambda t)^n}{n!} + \dots\right)v$$

$$= \underline{e^{\lambda t}v}$$

w $\underline{(A - \lambda I)^k w = 0}$

$$e^{tA}w = e^{t(A-\lambda I) + t\lambda I}w$$

$$= e^{t\lambda} e^{t(A-\lambda I)} w$$

$$= e^{t\lambda} \left(w + t(A-\lambda I)w + \frac{t^2}{2!}(A-\lambda I)^2 w + \dots \right)$$

$$= e^{t\lambda} \left(w + t(A-\lambda I)w + \dots + \frac{t^{k-1}}{(k-1)!}(A-\lambda I)^{k-1} w \right)$$

Example Exercise 26

$$y' = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix} y = Ay$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} -2-\lambda & 1 & -1 \\ 1 & -3-\lambda & 0 \\ 3 & -5 & -\lambda \end{pmatrix}$$

$$= \det \begin{pmatrix} -2-\lambda & 1 & -1 \\ 1 & -3-\lambda & 0 \\ 3+2\lambda+\lambda^2 & -5-\lambda & 0 \end{pmatrix}$$

$$= -1 \cdot \det \begin{vmatrix} 1 & -3-\lambda \\ 3+2\lambda+\lambda^2 & -5-\lambda \end{vmatrix}$$

$$= (5+\lambda) + (-3-\lambda)(3+2\lambda+\lambda^2)$$

$$= 5+\lambda - 9 - 6\lambda - 3\lambda^2 - 3\lambda - 2\lambda^2 - \lambda^3$$

$$= -4 - 8\lambda - 5\lambda^2 - \lambda^3$$

$$p(\lambda) = 0$$

$$\lambda^3 + 5\lambda^2 + 8\lambda + 4 = 0$$

$$(\lambda + 1)(\lambda^2 + 4\lambda + 4) = 0$$

$$(\lambda + 1)(\lambda + 2)^2 = 0$$

$$\lambda = -1 \quad \text{or} \quad \lambda = -2$$

$\lambda = -1$

$$(A + I)v = 0$$

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -2 & 0 \\ 3 & -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} -x + y - z \\ x - 2y \\ 3x - 5y + z \end{pmatrix} = 0$$

$$x = 2y$$

$$-y - z = 0 \quad \Rightarrow \quad y = -z$$

$$v_{-1} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = -2$

$$(A + 2I)\,v = 0$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{pmatrix} y - z \\ \end{pmatrix} = 0$$

$$\begin{pmatrix} x - y \\ 3x - 5y + 2z \end{pmatrix} = 0$$

$$y = z$$

$$x = y$$

$$v_{-2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(A + 2I)\, w = v_{-2}$$

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$(A+2I)^2 w = (A+2I)(A+2I) w$$

$$= (A + 2I)\, v_{-2}$$

$$= 0$$

$$(A+2I)^2 w = 0$$

$$\begin{pmatrix} y-z \\ x-y \\ 3x-5y+2z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$y = z+1$$

$$x = y+1 = z+2$$

$$(A+2I)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 3 & -5 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$w = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$e^{tA} v_{-1} = e^{-t} v_{-1}$$

$$= e^{-t} \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$e^{tA} v_{-2} = e^{-2t} v_{-2}$$

$$= e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$e^{tA} w = e^{-2t} \left(w + t(A+2I) w \right)$$

$$= e^{-2t} \left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$= e^{-2t} \begin{pmatrix} 2+t \\ 1+t \\ t \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 2+t \\ 1+t \\ t \end{pmatrix}$$

$$y(t) = C_1 e^{-t}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_3 e^{-2t}\begin{pmatrix} 1+t \\ t \end{pmatrix}$$