One more nullcline example:

$$\frac{dx}{dt} = x + y - x^3$$

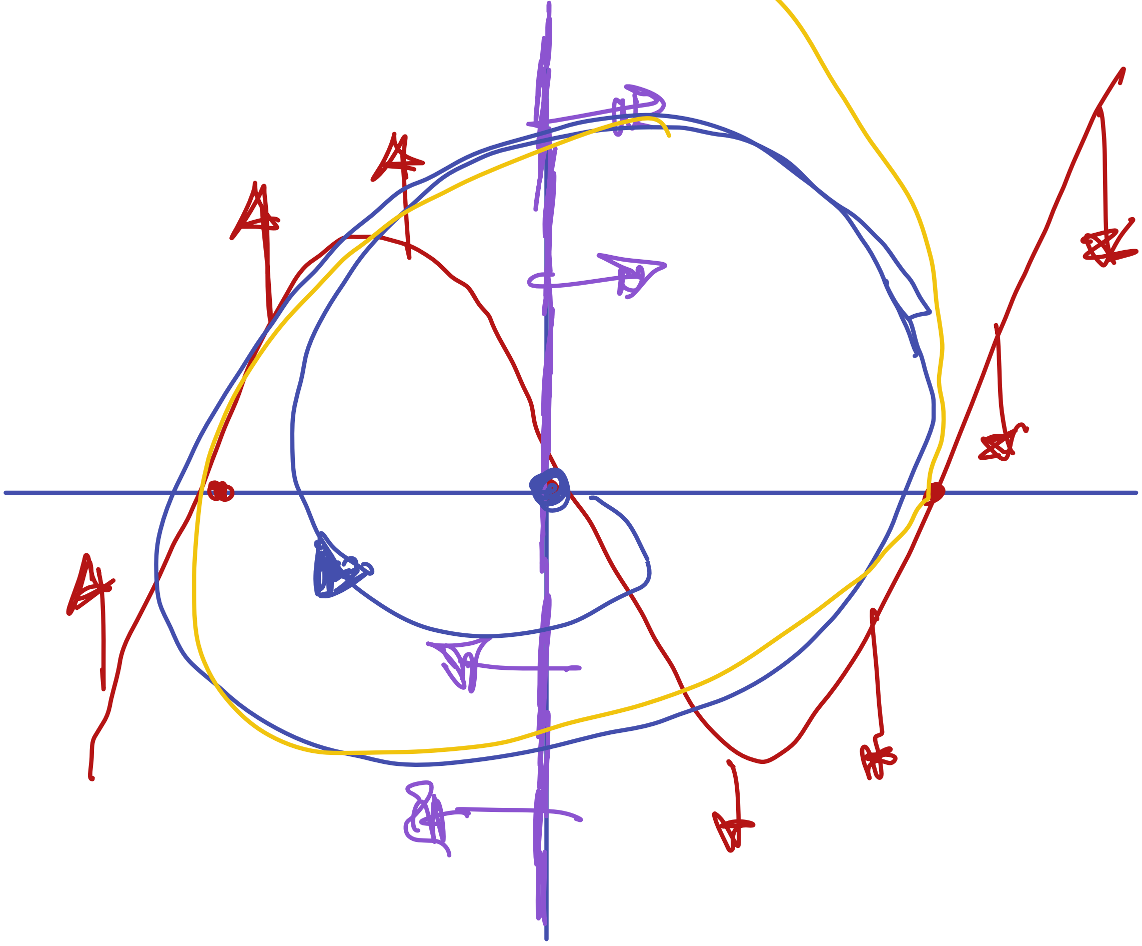
$$\frac{dy}{dt} = -\frac{1}{2}x$$

x-nullcline: $y = x^3 - x$

y-nullcline $x = 0$

EQ points: $(0,0)$

Let's get a sketch of trajectories.



looks like trajectories rotate around cw.

If we linearize the system:

$$J = \begin{pmatrix} 1-3x^2 & 1 \\ -1/2 & 0 \end{pmatrix}$$

$J(0,0)$

$$= \begin{pmatrix} 1 & 1 \\ -1/2 & 0 \end{pmatrix} \quad \begin{aligned} Tr &= 1 \\ Det &= 1/2 \end{aligned}$$

Spiral Source.

Since this is generic, the non-linear system has same behavior near $(0,0)$.

(If you actually plot trajectories, look like they converge to a "circular" path: this is something called a limit cycle, but we won't talk about these)

Application 1: Predator-Prey Model

Two species that exist and interact with each other: one a predator, one prey.

$F(t)$ = Prey
$S(t)$ = Predator

$$F' = r_F F$$
$$S' = r_S S$$

r_F, r_S reproduction rates.

If we assume there is

ample food in the absence of predators, $r_F = a > 0$.

Otherwise, when predators exist, each encounter with a predator carries some probability of death.

Encounters should be roughly proportional to # predators

$$r_F = a - bS \qquad a, b > 0.$$

Similarly without prey, the should die out. Encounters with the prey should increase reproductive rate prop. to # encounters

$r_S = -c + dF \qquad c, d > 0$

$F' = (a - bS)F$

$S' = (-c + dF)S$

Lokta-Volterra Model

There are two EQ points for the model:

$(0,0)$ and $\left(\frac{c}{d}, \frac{b}{a}\right)$

The Jacobian is $\begin{pmatrix} a-bS & -bF \\ dS & -c+dF \end{pmatrix}$

at $(0,0)$:

$\begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}$ <u>Saddle</u>

at $\left(\frac{c}{d}, \frac{b}{a}\right)$

$\begin{pmatrix} 0 & -bc/d \\ ad/b & 0 \end{pmatrix}$ <u>Center</u>

Since a center is non-genric, we need some other way of determining what's happen in the system at this point.

One strategy is as follows:

Heuristically, we think that if both populations are non-zero they should behave periodically. This would mean this EQ pt should be a center.

Trajectories around a center have to be a closed curve.

Idea: Find $H(F,S)$ s.t.

$$\frac{d}{dt} H(F(t), S(t)) = 0$$

this would mean $(F(t), S(t))$ are points on some level curve of H, and level curves are generally closed.

$$\frac{d}{dt} H(F,S)$$

$$= \frac{\partial H}{\partial F} \cdot \frac{dF}{dt} + \frac{\partial H}{\partial S} \cdot \frac{dS}{dt} = 0$$

$$\frac{\partial H}{\partial F}(a-bS)F + \frac{\partial H}{\partial S}(-c+dF)S = 0$$

Note that $\frac{dF}{dS} = \frac{dF/dt}{dS/dt} = \frac{(a-bS)F}{(-c+dF)S}$

this is separable:

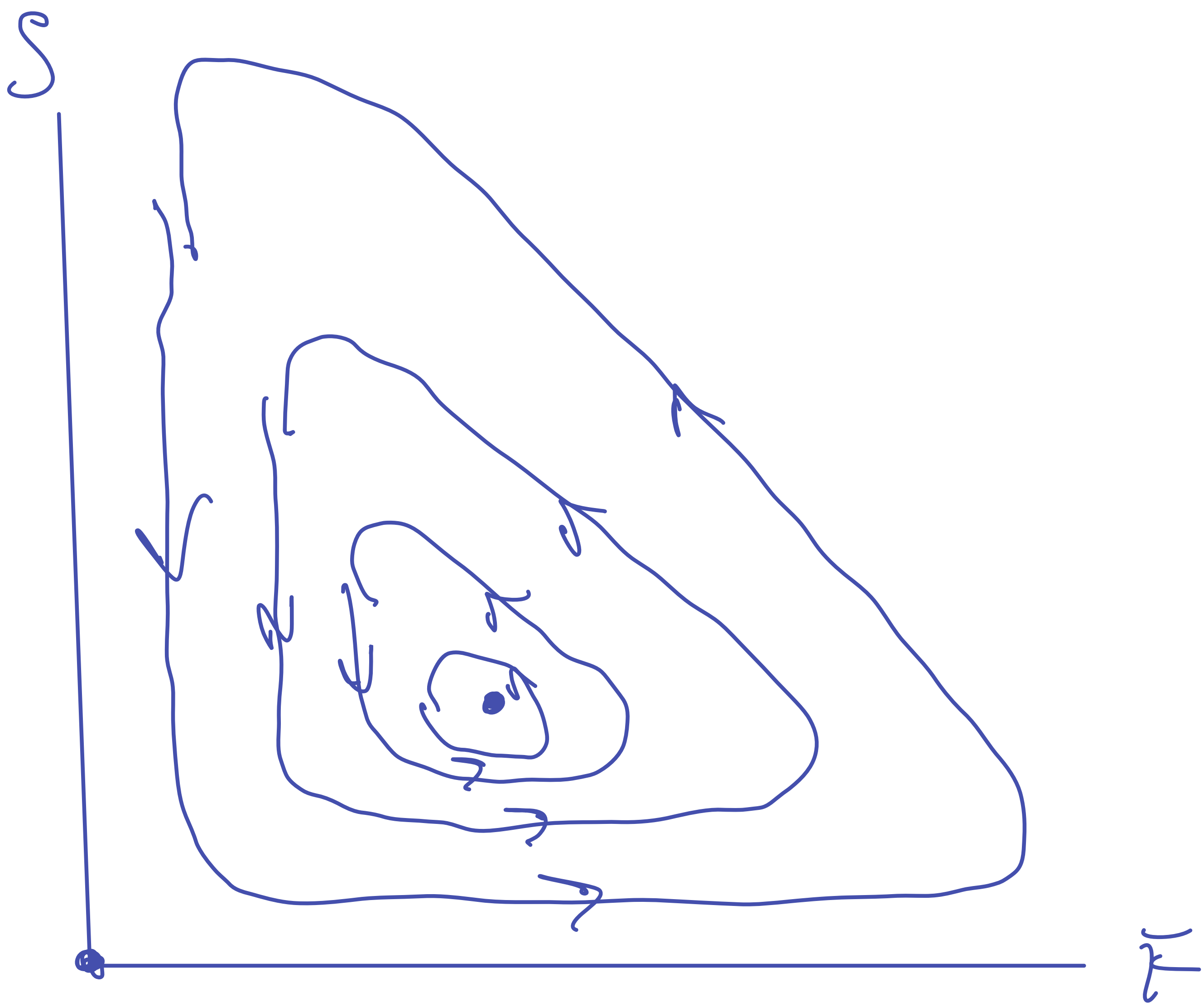
$$\frac{-c+dF}{F}\,dF = \frac{a-bS}{S}\,dS$$

$$\Rightarrow \quad bS - a\ln(S) + dF - c\ln(F) = C$$

$$H(F,S) = bS - a\ln(S) + dF - c\ln(F)$$

Satisfies the above eq^n $\Rightarrow$

trajectories are level curves of H. One can show that level curves of H are closed, $\Rightarrow$ that (x_0, y_0) is indeed a center



Remark: the picture on the course page is this!

Application 2: Competing Species

Now, suppose we have two species that compete for the same kind of resource. Assume in the absence of one, the other species grows logistically

$$\frac{dR}{dt} = K_1 R\left(1 - \frac{R}{L_1}\right)$$

$$\frac{dS}{dt} = K_2 S\left(1 - \frac{S}{L_2}\right)$$

the simplest model for how they should interact should be when we assume the per capita growth rate dec.

linearly with density of other species:

$$\frac{dR}{dt} = K_1 R\left(1 - \frac{R}{L_1} - \frac{c_1}{K_1} S\right)$$ for some $c_1, c_2 > 0.$

$$\frac{dS}{dt} = K_2 S\left(1 - \frac{S}{L_2} - \frac{c_2}{K_2} S\right)$$

Simplified, we get equations that look like the following:

$$\boxed{\begin{aligned} \frac{dR}{dt} &= K_1 R\left(1 - \frac{R}{L_1}\right) - c_1 RS \\ \frac{dS}{dt} &= K_2 S\left(1 - \frac{S}{L_2}\right) - c_2 RS \end{aligned}}$$

you can think of the constants c_1, c_2 as some parameters that measure how strong the competition between species is.

Ex: $$\frac{dR}{dt} = R(1-R-S)$$

$$\frac{dS}{dt} = S\left(\frac{3}{4} - S - \frac{1}{2}R\right)$$

Critical points: $(0,0)$, $(0, 3/4)$, $(1,0)$, $(1/2, 1/2)$.

$1 - R - S = 0$

$\frac{3}{4} - S - \frac{1}{2}R = 0$

Jacobian:

$$\begin{pmatrix} 1-2R-S & -R \\ -\frac{1}{2}S & \frac{3}{4}-2S-\frac{1}{2}R \end{pmatrix}$$

At each Critical point:

$(0,0)$: $\begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix}$ Nodal Source

$(0,3/4)$: $\begin{pmatrix} 1/4 & 0 \\ -3/8 & -3/4 \end{pmatrix}$ Saddle

$(1,0)$: $\begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix}$ Saddle

$(\frac{1}{2},\frac{1}{2})$: $\begin{pmatrix} -1/2 & -1/2 \\ -1/4 & -1/2 \end{pmatrix}$ Nodal Sink

$T = -1$

$D = 1/8$

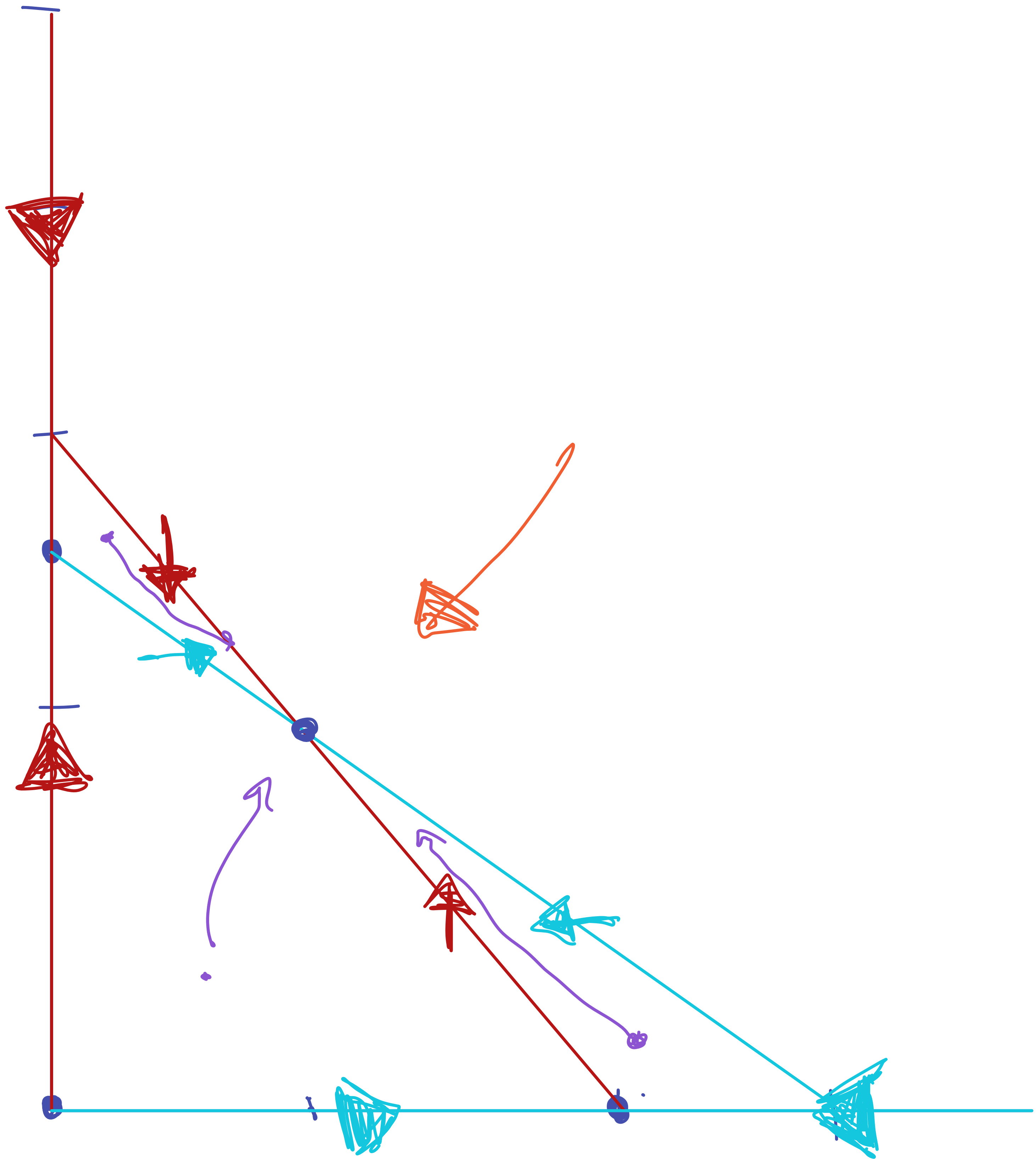
Global Picture:

R-nullcline: $R = 0$ or

$S = 1 - R$

S-nullcline: $S = 0$ or

$S = \frac{3}{4} - \frac{1}{2} R$



$$\frac{dR}{dt} = R(1-R-S)$$

$$\frac{ds}{dt} = S\left(\tfrac{3}{4} - S - \tfrac{1}{2}R\right)$$

$$(1-R)\cdot\left(\tfrac{3}{4} - (1-R) - \tfrac{1}{2}R\right)$$

$$(1-R)\left(-\tfrac{1}{4} + \tfrac{1}{2}R\right)$$

$$\left(\tfrac{3}{4} - \tfrac{1}{2}R\right)\left(1 - R - \left(\tfrac{3}{4} - \tfrac{1}{2}R\right)\right)$$

$$\left(\tfrac{3}{4} - \tfrac{1}{2}R\right)\left(\tfrac{1}{4} - \tfrac{1}{2}R\right)$$

All initial conditions get pulled towards $(1/2, 1/2)$, so the species are able to coexist!

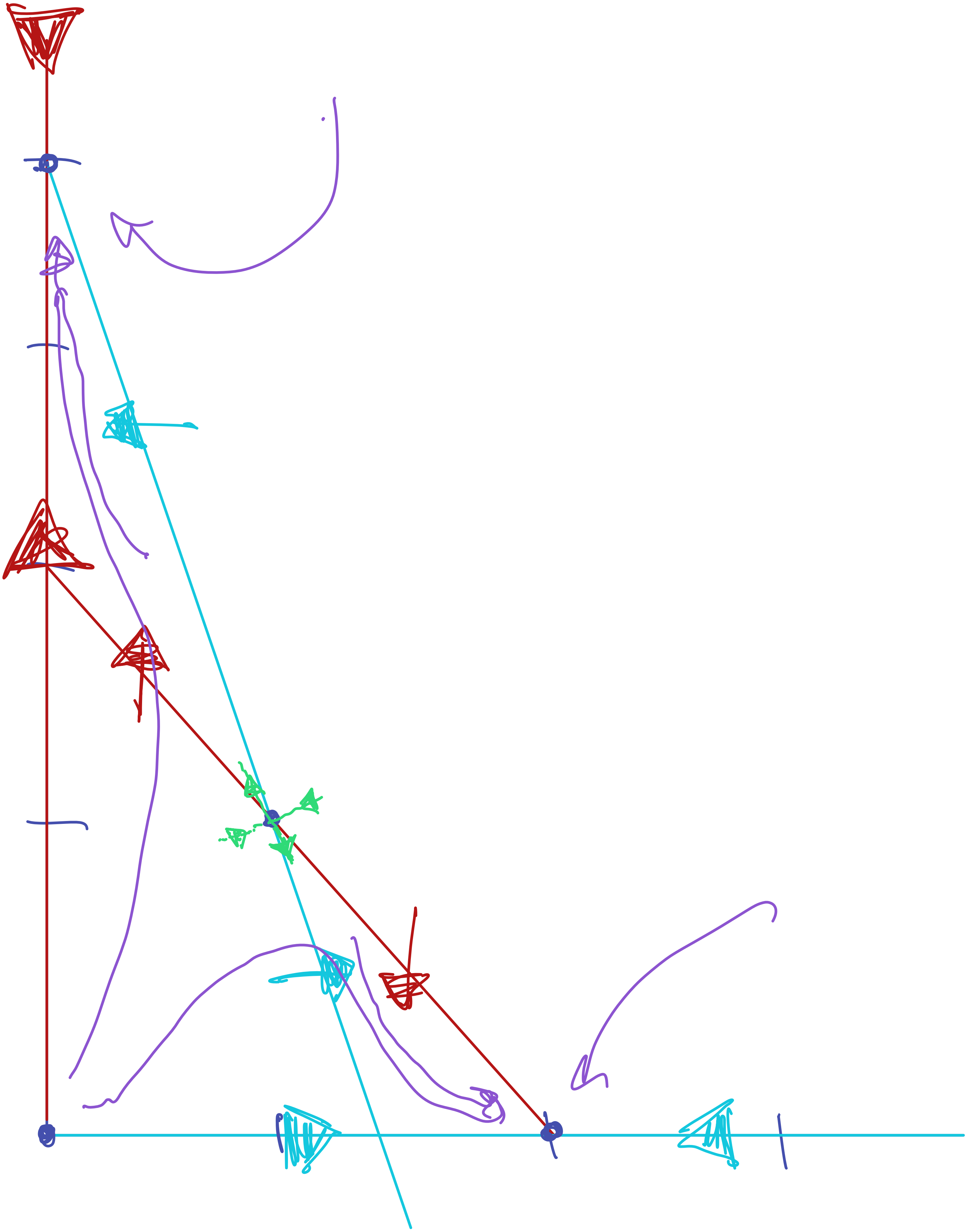
Ex:

$$\frac{dR}{dt} = R(1-R-S)$$

$$\frac{dS}{dt} = S\left(\frac{1}{2} - \frac{1}{4}S - \frac{3}{4}R\right)$$

EQ points: $(0,0)$, $(1,0)$, $(0,2)$, $(1/2, 1/2)$.

Let's do a global analysis using nullclines:



We tend to either $(1,0)$ or $(0,2)$ and so a species must always die.

At $(1/2, 1/2)$:

$$J = \begin{pmatrix} 1-2R-S & -R \\ -\frac{3}{4}S & \frac{1}{2}-\frac{1}{2}S-\frac{3}{4}R \end{pmatrix}$$

$$\begin{pmatrix} -1/2 & -1/2 \\ -3/8 & 1/8 \end{pmatrix}$$ Saddle

If we sketch the eigenvectors we see this is why sol^{n}s can never tend to $(1/2, 1/2)$.

Application 3: SIR

Disease spreading model.

S = # susceptible individuals

I = # Infected individuals

R = # Removed individuals

$N =$ Population size

$S + I + R = N \Rightarrow$

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

Assumptions:

- Infected individuals have b contacts per day w/ susceptible individuals.

So each infected individual generates

$\sim \frac{b}{N} S$ infections, so

$$\frac{dS}{dt} = -\frac{b}{N} S I$$

Now, assume some fraction k of Infected individuals are removed during any day.

$$\frac{dR}{dt} = \frac{k}{N} I$$

$$\frac{dI}{dt} = \frac{b}{N} I S - \frac{k}{N} I$$

SIR has been used to model the spread

of COVID-19, for example.

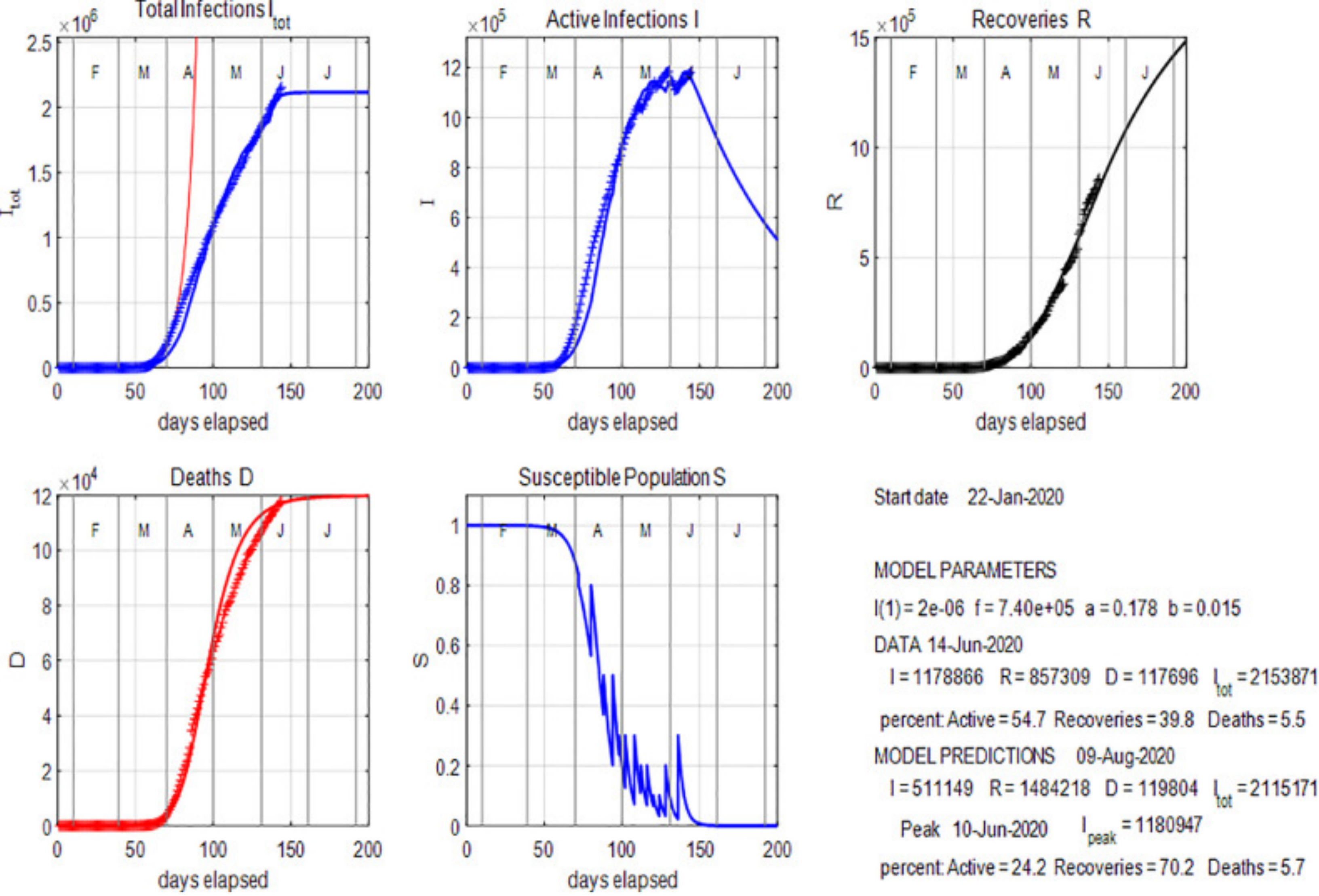
Some data:

1/22/2020 $\longrightarrow$ 6/14/2020

$b/N \approx .178$ $\qquad$ $k/N \approx .015$

$I(1) = 2\times 10^{-6}$ $\qquad$ $0,$

(t days, I was measured in millions)



Why does SIR say that Infections follow a bell curve shape?

$$\frac{dI}{dt} = \frac{b}{N} SI - \frac{k}{N} I$$

at start, $\frac{S}{N} \approx 1$

$$\frac{dI}{dt} \simeq \left(b - \frac{k}{N}\right) I$$

eventually,

$$\frac{dI}{dt} = 0 \qquad \frac{b}{N} S - \frac{k}{N} = 0.$$

After which point, $\frac{b}{N} S - k/N < 0$

So should decay exponentially

(when $S \simeq 0$ looks like $\frac{dI}{dt} \simeq -k/N\, I$)