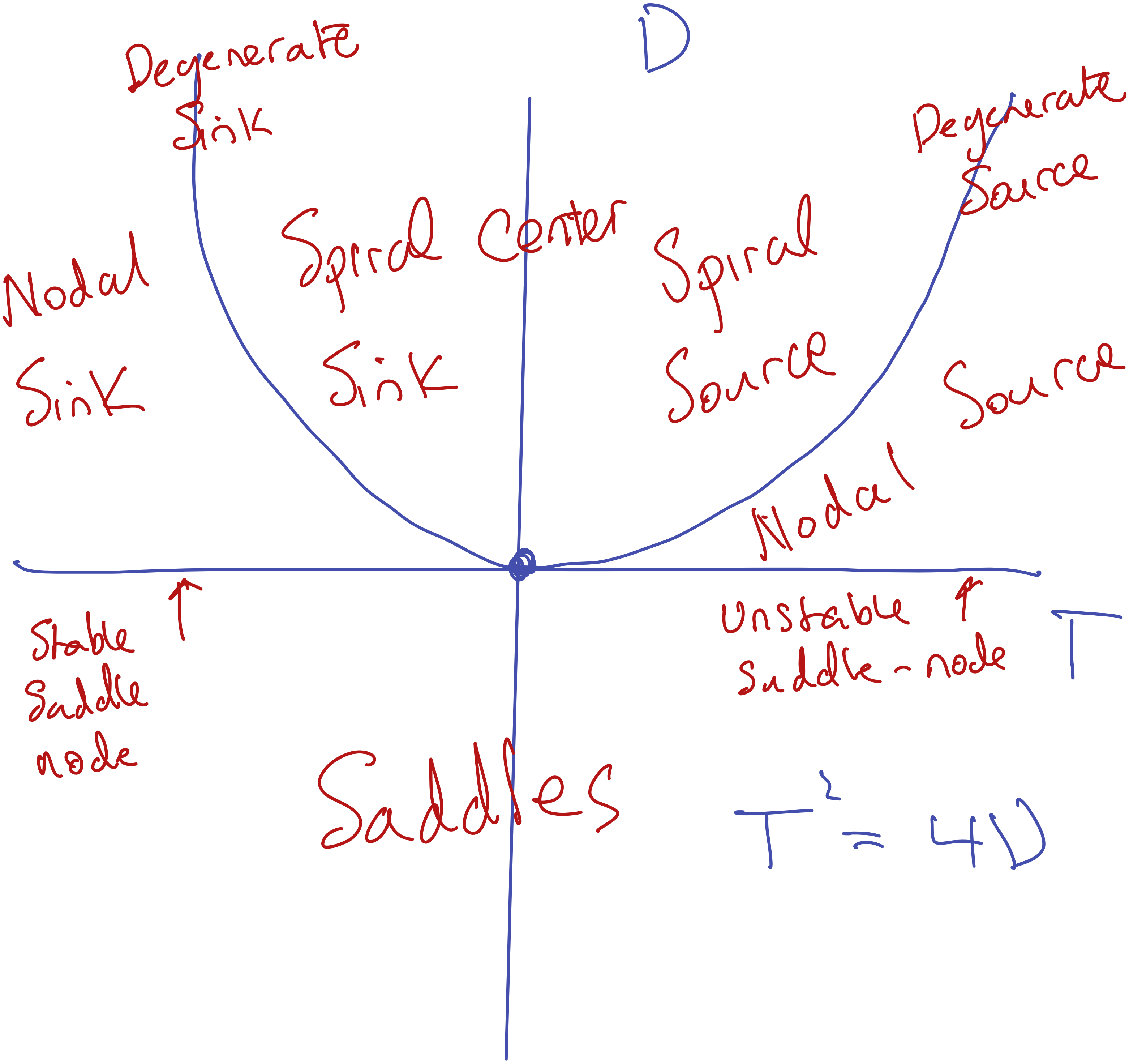
Last time: classified equilibrium behavior for 2x2 system.

$A \qquad p_A(x) = x^2 - Tr(A)x + det(A)$

$$\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

Remember: most of the classification was done based on eigenvalues.

Trace - Determinant Plane:



Trace-Determinant plane is a useful example of something called a parameter space.

$Y' = AY$

Examples

- $A = \begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix}$ $\quad T = 6$, $D = 2$

$\Longrightarrow$ Nodal Source

- $A = \begin{pmatrix} 5 & -3 \\ -8 & -6 \end{pmatrix}$ $\quad T = -1$, $D = -6$

$\Longrightarrow$ Saddle

- $A = \begin{pmatrix} 1 & -3 \\ 3 & -5 \end{pmatrix}$ $\quad T = -4$, $D = 4$

$\Longrightarrow$ degenerate nodal Sink.

· $A = \begin{pmatrix} 8 & 20 \\ -4 & -8 \end{pmatrix}$ $\quad T = 0$, $D = 16$

$\Rightarrow$ Center.

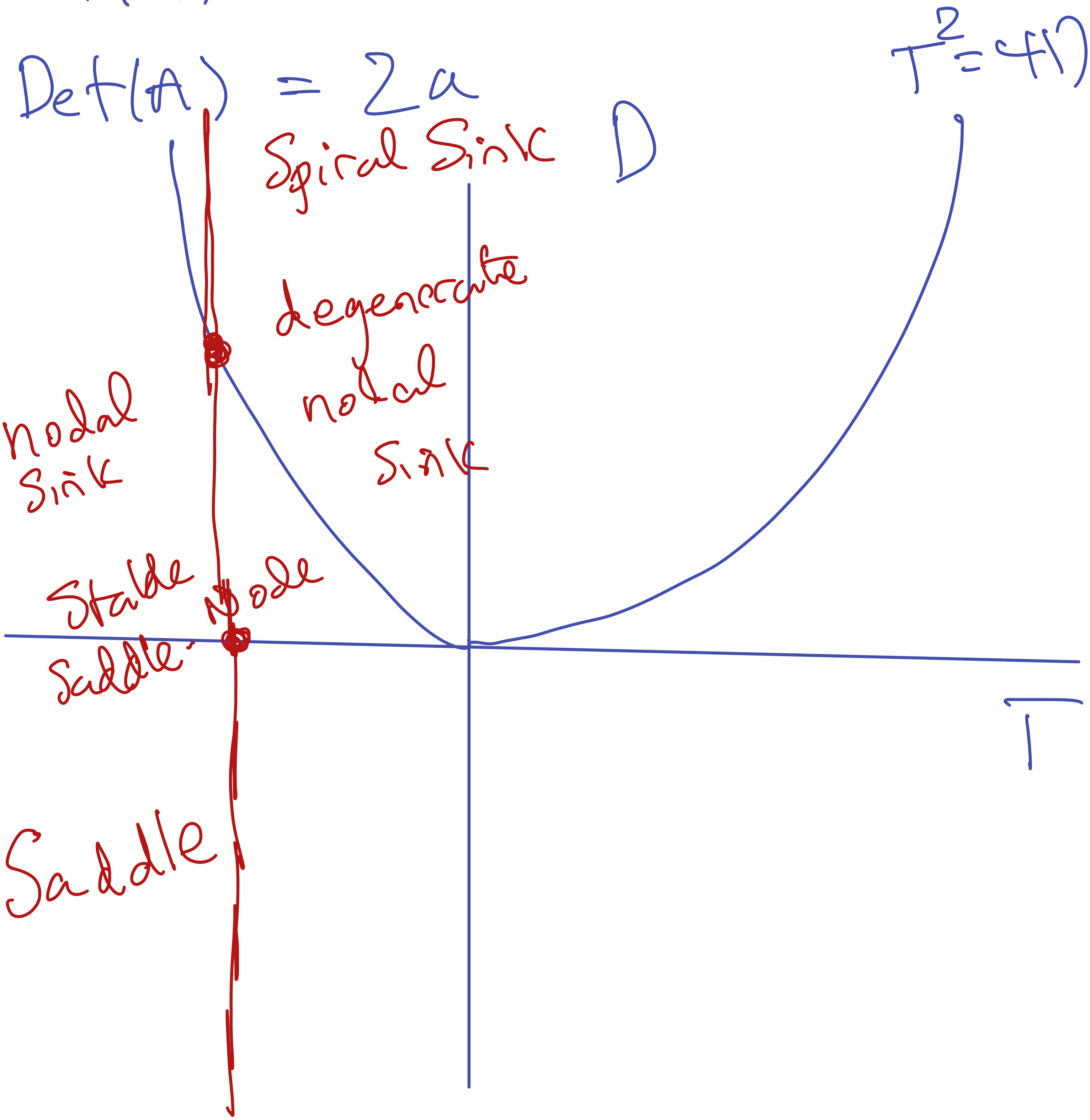
Bifurcation:

Let's examine a family of linear systems of ODEs and see how the behavior changes depending on a parameter.

$$Y' = AY \qquad A = \begin{pmatrix} -2 & a \\ -2 & 0 \end{pmatrix}$$

$$Tr(A) = -2$$

$$Det(A) = 2a$$



As we vary a, the system we get lives on the red line in the T-D plane.

bifurcation happens the red dots:

$$\text{Det}(A) = 0 \implies 2a = 0$$
$$\implies a = 0$$

$$T^2 = 4D \implies 4 = 8a$$
$$\implies a = 1/2.$$

Two bifurcation values:

$a = 0, 1/2.$

Let's investigate the bifurcation at $1/2$ a little more closely.

$$p_A(x) = x^2 + 2x + 2a$$

has roots

$$\lambda = \frac{-2 \pm \sqrt{4 - 8a}}{2}$$

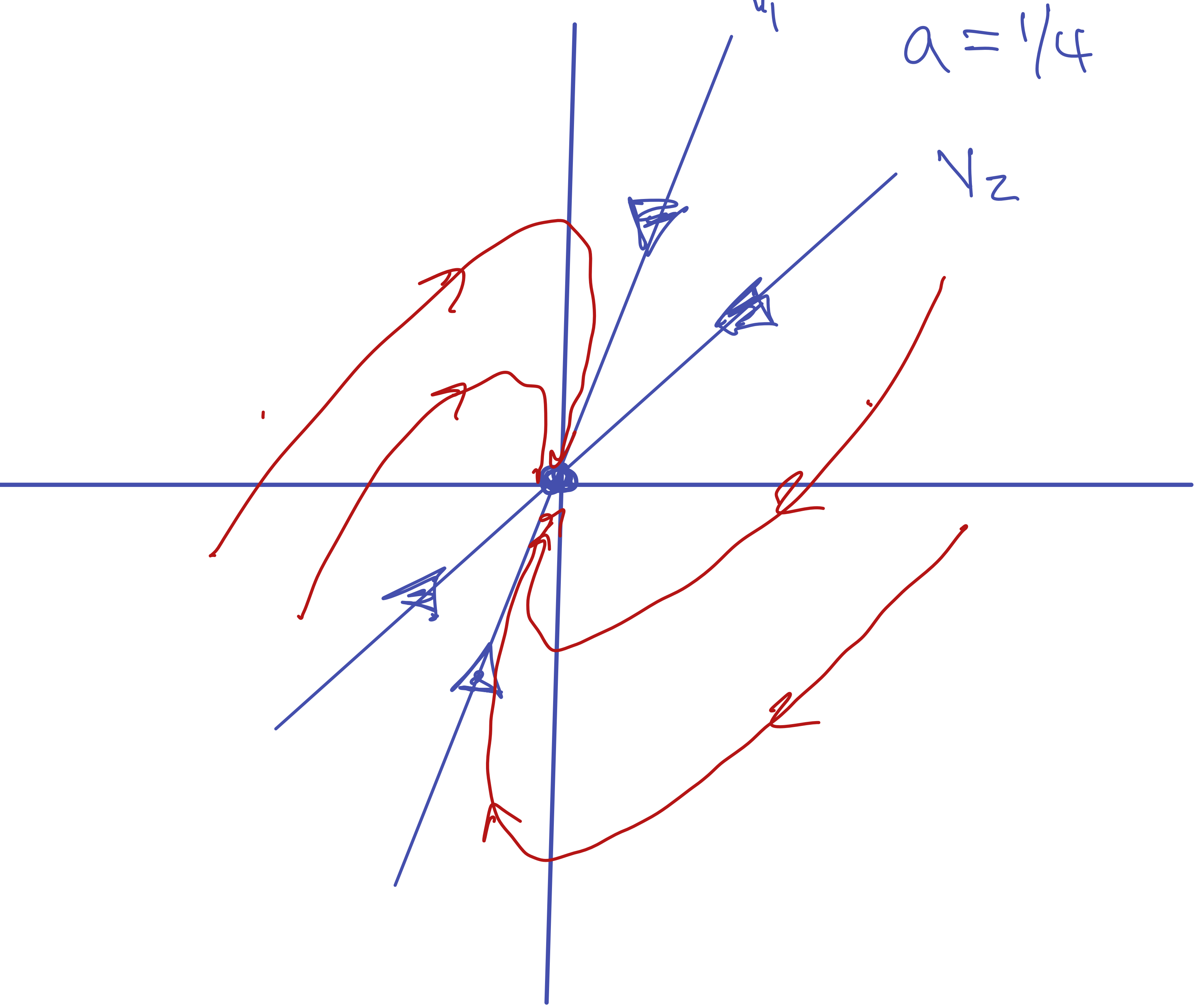
$$= -1 \pm \sqrt{1 - 2a}.$$

The eigenvectors associated to each eigenvalue are as follows:

$$\lambda_1 = -1 + \sqrt{1-2a} \qquad v_1 = \begin{pmatrix} a \\ 1+\sqrt{1-2a} \end{pmatrix}$$

$$\lambda_2 = -1 - \sqrt{1-2a} \qquad v_2 = \begin{pmatrix} a \\ 1-\sqrt{1-2a} \end{pmatrix}$$

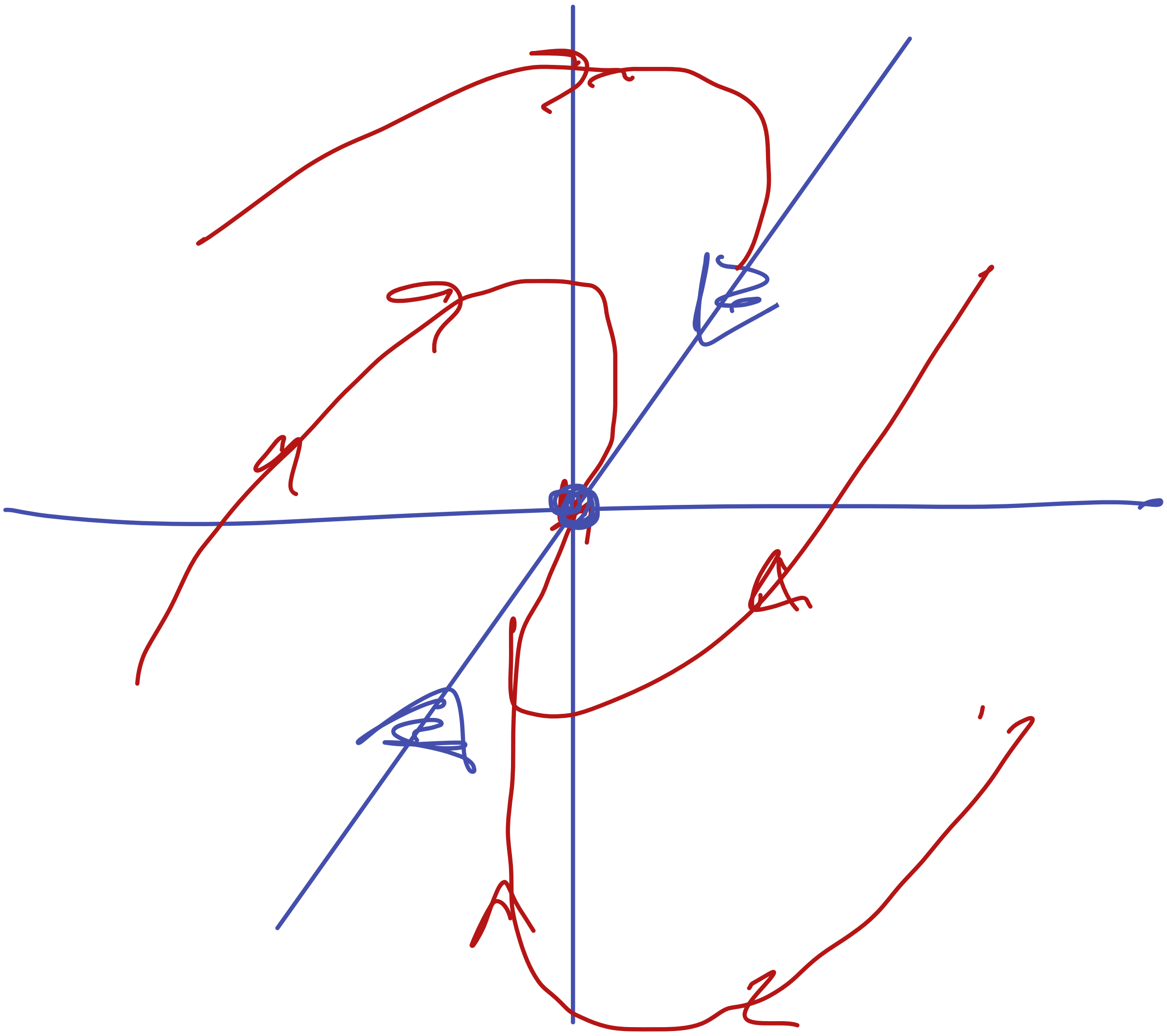
$a = 1/4$



$v_1 : \quad 4(1+\sqrt{1/2})\,x$

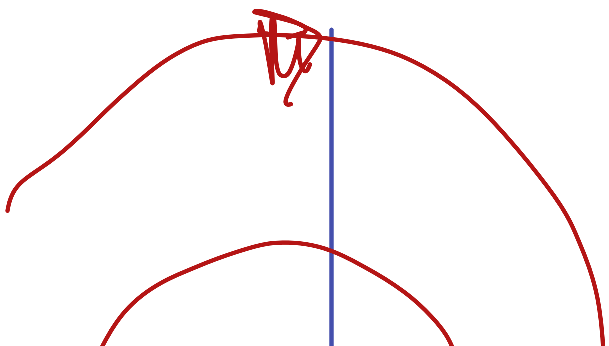
$v_2 : \quad 4(1-\sqrt{1/2})\,x$

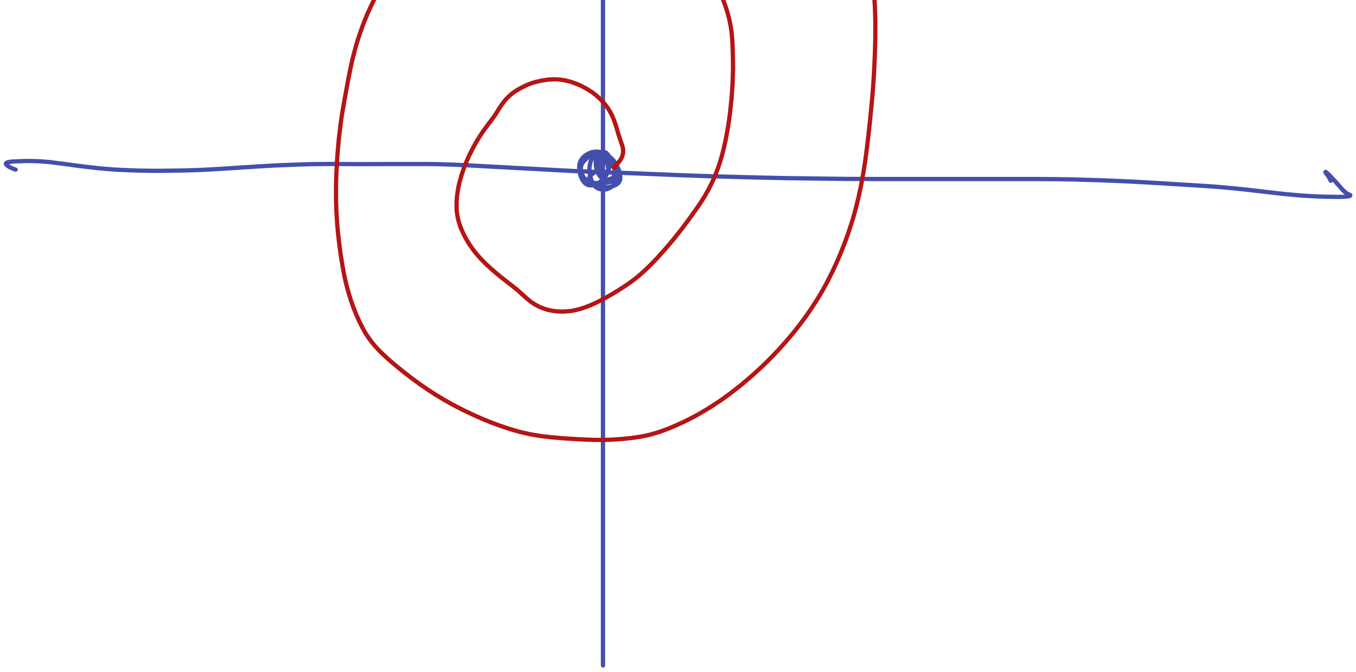
$a = 1/2$



$V: \; 2x$

$a = 5$





Another example:

Harmonic Oscillation.

$$my'' + by' + ky = 0$$

For the sake of example, will pick $m=1, k=3$.

How the behavior change as we vary the dampening factor b?

$(b \geq 0.)$

First, let's convert to a system:

$$y' = v$$

$$y'' = -by' - 3y = v'$$

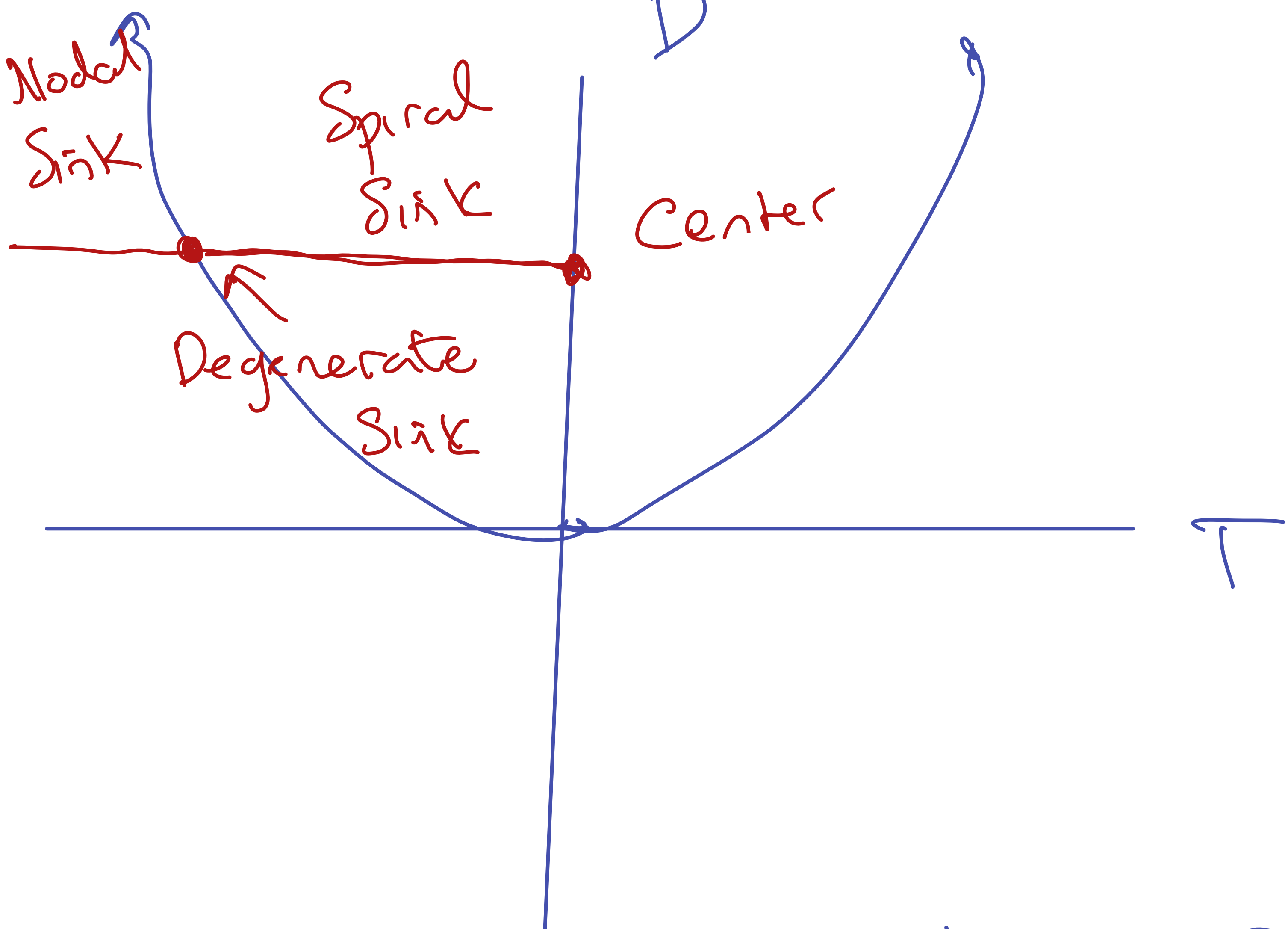
$$= -bv - 3y$$

$$y' = v$$

$$v' = -3y - bv$$

$$Y' = \begin{pmatrix} 0 & 1 \\ -3 & -b \end{pmatrix} Y$$

$\mathrm{Tr}(A) = -b$ $\quad\quad$ Det $= 3$



bifurcation at $b = 0, 2\sqrt{3}$

$b^2 = 12 \implies b = 2\sqrt{3}$

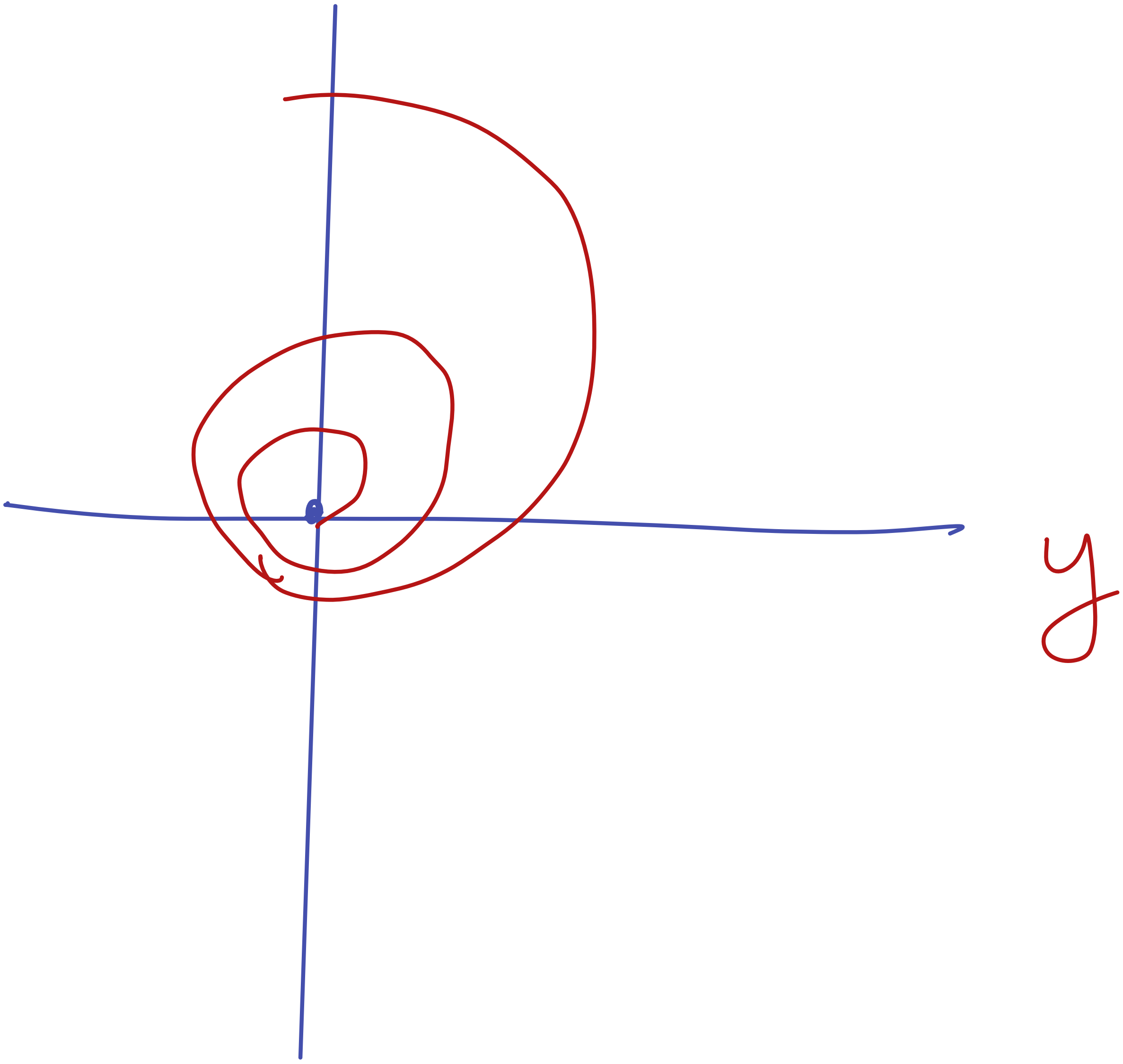
if we plug in $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ -3 & b \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

So all rotations will be clockwise.

- $b = 0 \Rightarrow$ cw center
- $0 < b < 2\sqrt{3}$ cw spiral sink
- $b = 2\sqrt{3}$ Degenerate sink
- $b > 2\sqrt{3}$ Nodal sink

✓



Center: undamped

oscillates infinitely

Spiral Sink: underdamped

decaying oscillation

to 0

Degen Sink: Critically damped: no oscillation goes back to starting pos. as fast as possible and stops.

Nodal Sink: overdamped block slowly goes back to starting position.

Runge-Kutta

As you saw on HW 3, Euler's method is not numerically stable.

Let's talk about a better method.

Technically, these are a family of methods but "Runge-Kutta" usually means "RK4".

The idea: If you remember numerical integration from calc:

Eulers method $\sim$ Trapezoidal Rule

RK $\sim$ Simpson's

We're going to construct a weighted average of slopes to make a step.

t_k $\quad$ y_k $\quad$ Δt

- Four Slopes: m_k, n_k, p_k, q_k

- $m_k = f(t_k, y_k)$

- $\tilde{t} = t_k + \frac{\Delta t}{2}$

$$\tilde{y}_k = y_k + m_k \frac{\Delta t}{2}$$

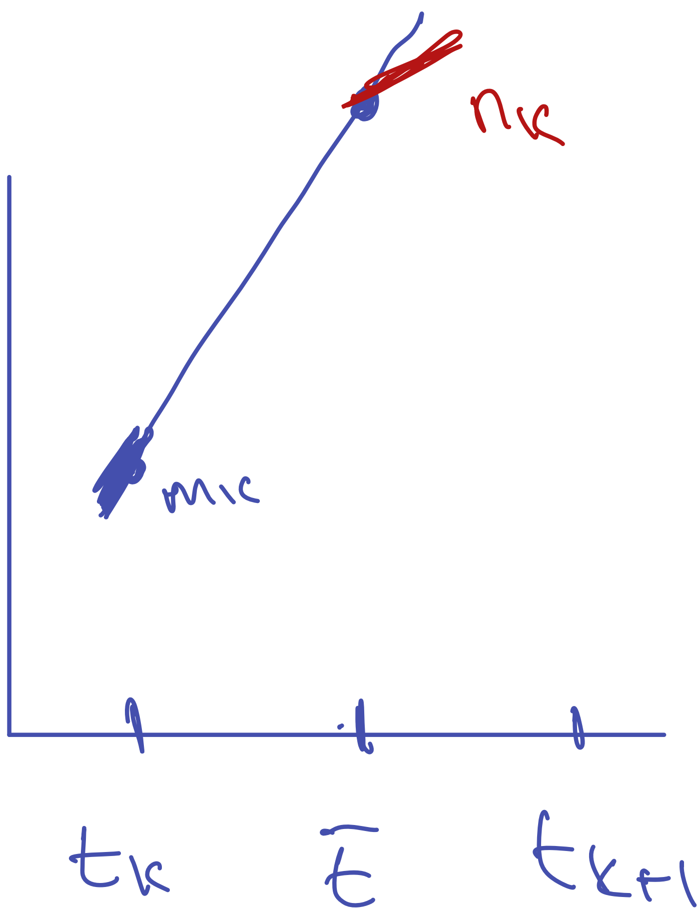
$$n_k = f(\hat{t}, \tilde{y}_k).$$

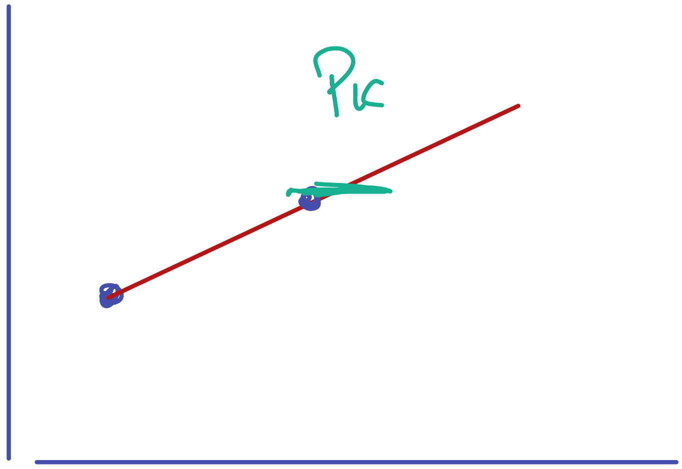
$$\hat{y}_k = y_k + n_k \cdot \frac{\Delta t}{2}$$

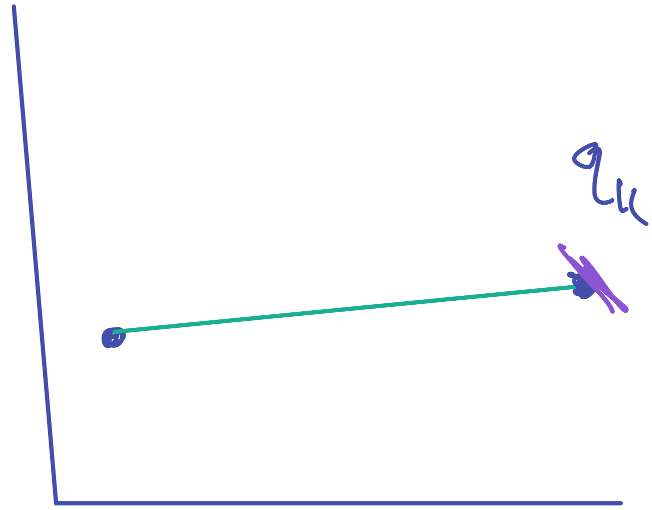
$$q_k = f(\hat{t}, \hat{y}_k)$$

$$\bar{y}_k = y_k + q_k \Delta t$$

$$p_k = f(t_{k+1}, \bar{y}_k).$$







The step is given by

$$y_{k+1} = y_k + \frac{1}{6}(m_k + 2n_k + 2q_k + p_k)\Delta t$$

$$t_{k+1} = t_k + \Delta t$$

Ex: $y' = y - t \qquad y(1) = 1$

Run 4 iterates to estimate

$y(1.4)$.

First Step: $y_0 = 1 \qquad t_0 = 1$

$\hat{t} = 1.05$

$m_0 = f(1,1) = 1-1 = 0$

$n_0 = f(\tilde{t}, y_0 + m_0 \cdot .05) = 1 - 1.05$
$= -.05$

$q_0 = f(\tilde{t}, y_0 + n_0 \cdot .05) = -.0525$

$p_0 = f(t_1, y_0 + q_0 \cdot .1) = -.10525$

$y_1 = y_0 + .1\left(\frac{\cdots}{6}\right) = .994829$

$t_1 = 1 + .1 = 1.1$

t_k	y_k (RK)	y_k (Euler)
1	1	1
1.1	.994829167	1
1.2	.978597429	.99

1.3	.950141502	.969
1.4	.908175719	.9359

t_k	Y_k (Actual)
1	1
1.1	.994829081
1.2	.978517241
1.3	.950141192
1.4	.908175302

Works way better!

Quick remark on error:

- Euler's method is a <u>first order</u> technique.

b/c it uses first order Taylor approx. (i.e. tangent lines).

Error at each step $\approx c \cdot h^2$ (Taylor's thm)

Total error $\approx$ # steps $ch^2 \approx \frac{1}{n} ch^2$

$\approx c \cdot h$ for some c.

- RK 4 is a <u>fourth order</u> method: there are 4th order Taylor polynomials

Somewhere in the background.

Error at each step $\approx c \cdot h^5$ (Taylor's thm)

Total Error $\approx c \cdot h^4$ for some c.

In actuality, with round off error it's roughly

$$\approx c_1 \cdot h^4 + \frac{c_2}{h},$$ So

Step size with RK shouldn't be too small to make sure round off error isn't more significant.