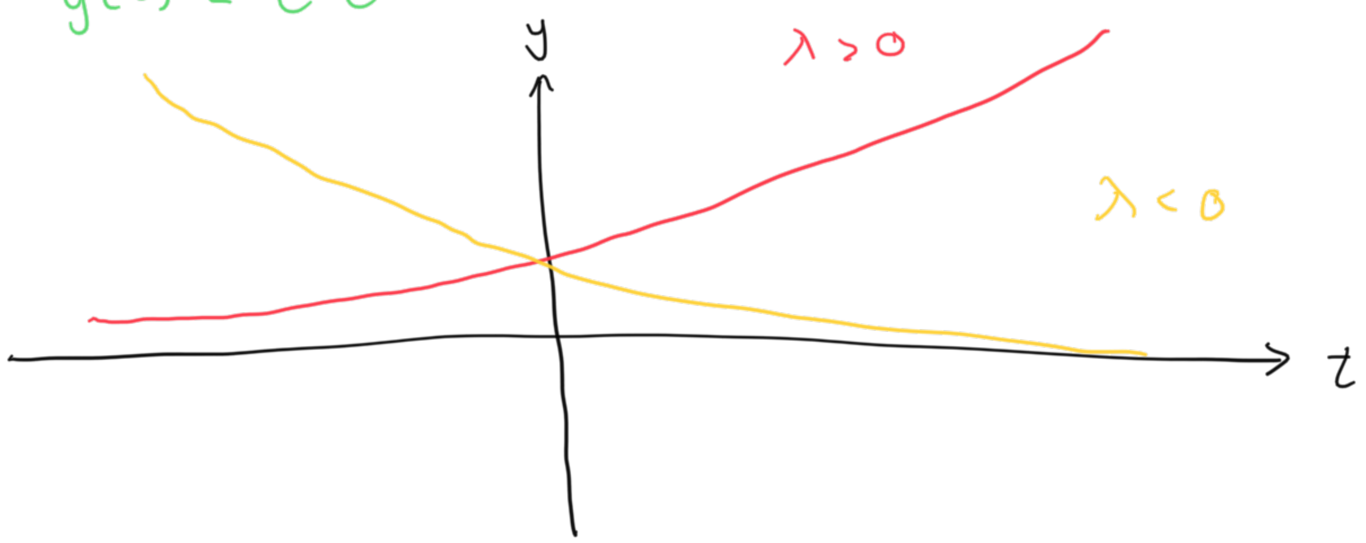
Math 33B : 7/19

Planar system & the stability of the solutions

$$y' = \lambda y$$

$$y(t) = C e^{\lambda t}$$



$$y' = Ay = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} y$$

$$T = a_{11} + a_{22}, \quad D = a_{11}a_{22} - a_{12}a_{21}$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

$$= \lambda^2 - T\lambda + D$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

$\lambda_1 \quad \lambda_2$

$v_1 \quad v_2$

$$y' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} y$$

$$p(\lambda) = \lambda^2 - 9 = (\lambda - 3)(\lambda + 3)$$

$$\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -3\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a + 4b = -3a$$

$$a + b = 0$$

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 3\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a + 4b = 3a$$

$$a = 2b$$

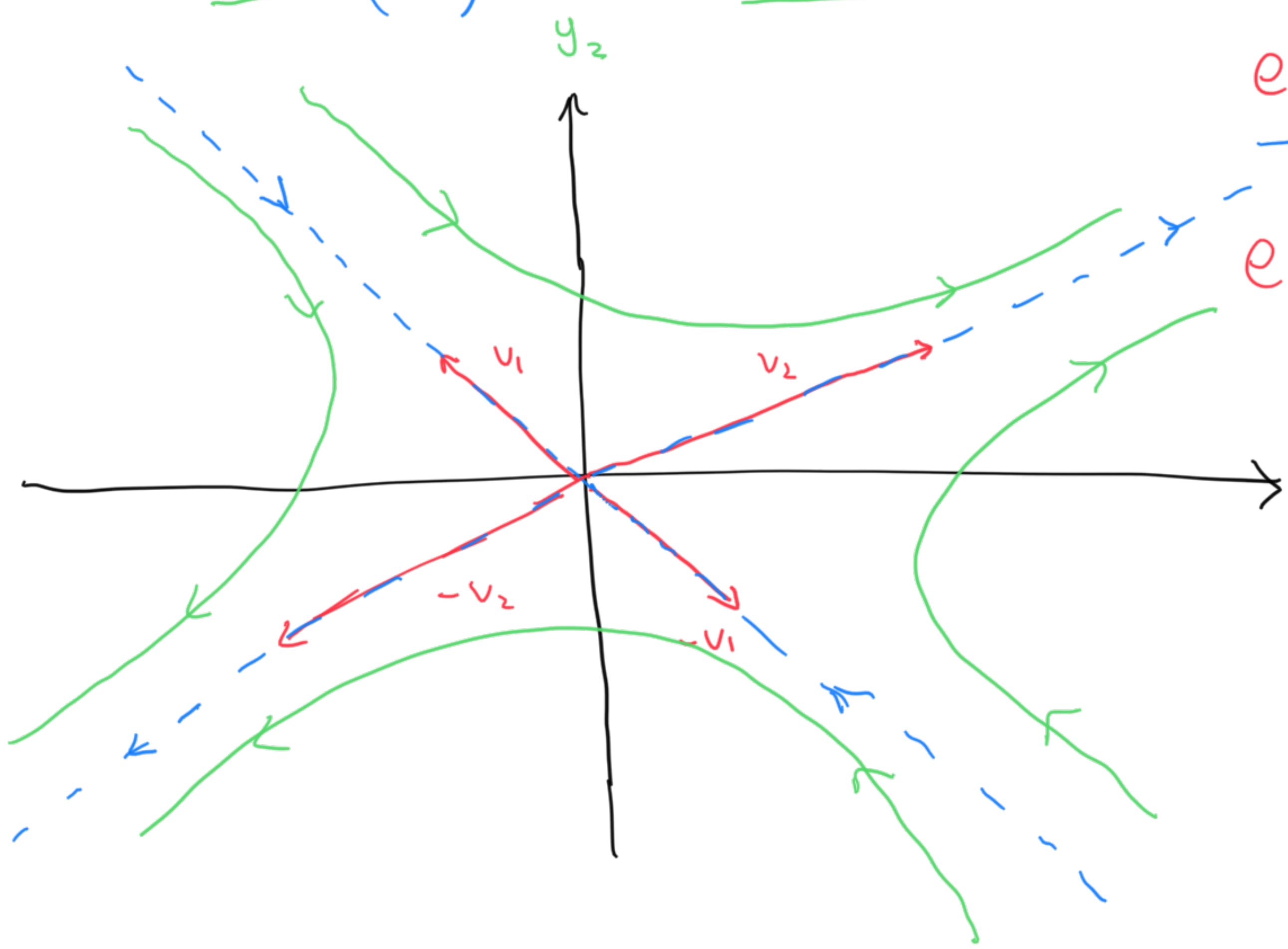
$$v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$y(t) = C_1 \underline{e^{-3t}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \underline{e^{3t}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

$$\underline{e^{-3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$$e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$



$\lambda_1 < 0 < \lambda_2$ saddle point

$$y' = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix} y$$

$$p(\lambda) = \lambda^2 + 6\lambda + 8 = (\lambda + 4)(\lambda + 2)$$

$$\begin{pmatrix}-3 & -1\\ -1 & -3\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = -4\begin{pmatrix}a\\ b\end{pmatrix}$$

$$-3a - b = -4a$$

$$a - b = 0$$

$$v_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\begin{pmatrix}-3 & -1\\ -1 & -3\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = -2\begin{pmatrix}a\\ b\end{pmatrix}$$

$$-3a - b = -2a$$

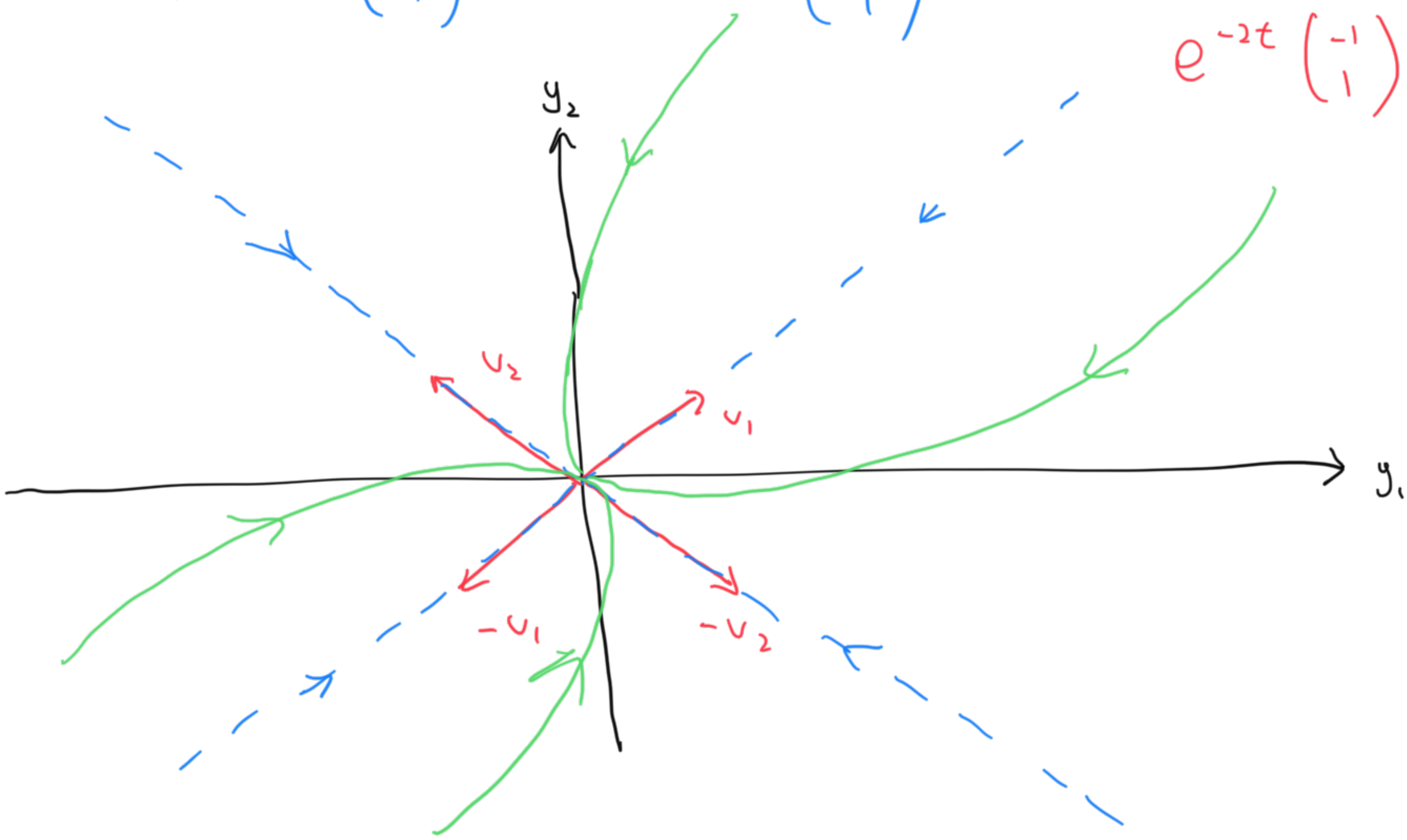
$$a = -b$$

$$v_2 = \begin{pmatrix}-1\\ 1\end{pmatrix}$$

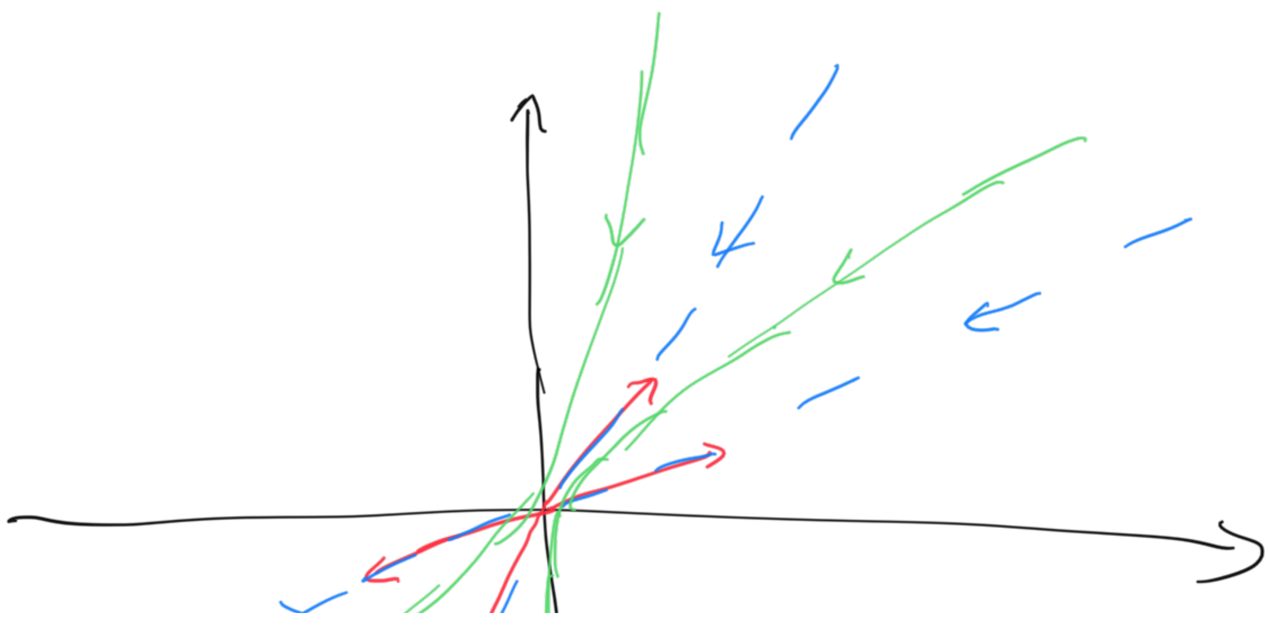
$$y(t) = C_1 e^{-4t}\begin{pmatrix}1\\ 1\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}-1\\ 1\end{pmatrix}$$

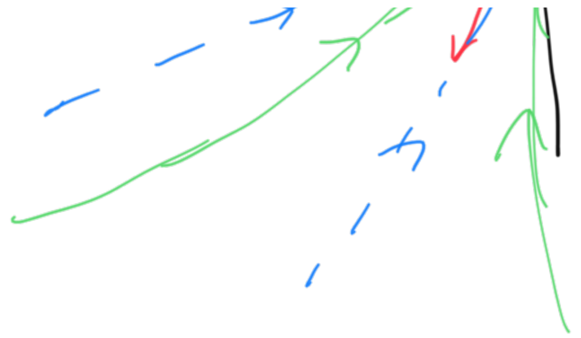
$e^{-4t}\begin{pmatrix}1\\ 1\end{pmatrix}$

$e^{-2t}\begin{pmatrix}-1\\ 1\end{pmatrix}$



$\lambda_1 < \lambda_2 < 0$ nodal sink





$$y' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} y$$

$$p(\lambda) = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix}$$

$$3a + b = 2a$$

$$a + b = 0$$

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

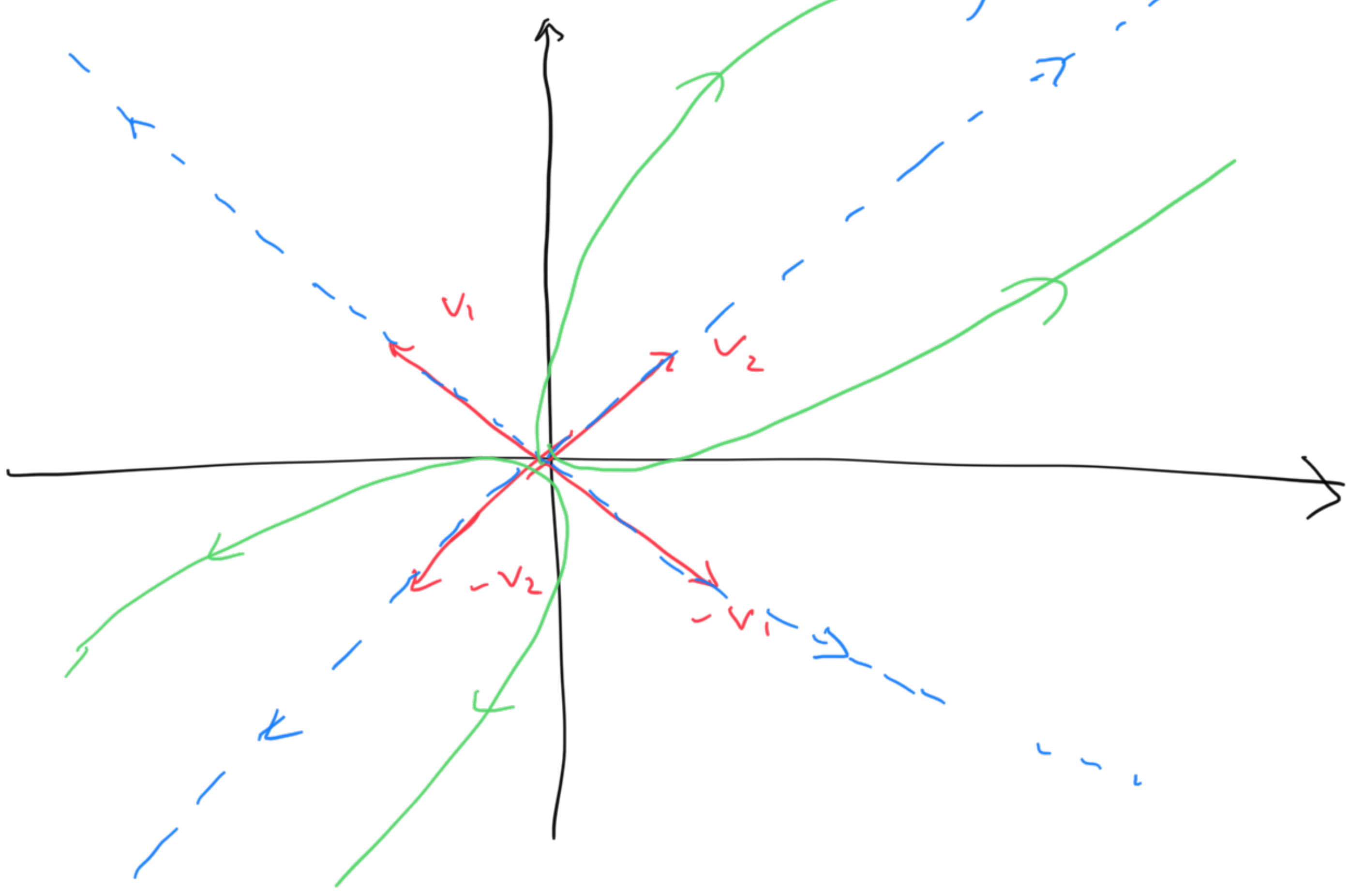
$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 4\begin{pmatrix} a \\ b \end{pmatrix}$$

$$3a + b = 4a$$

$$a = b$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$y(t) = c_1 e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



$0 < \lambda_1 < \lambda_2$ nodal source

$T^2 - 4D < 0$, T

$\lambda = \alpha + i\beta$ $e^{\alpha t}\left((\cos\beta t)v_1 - (\sin\beta t)v_2\right)$

$w = v_1 + i v_2$ $e^{\alpha t}\left((\sin\beta t)v_1 + (\cos\beta t)v_2\right)$

$$y' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} y$$

$$p(\alpha) = \lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i)$$

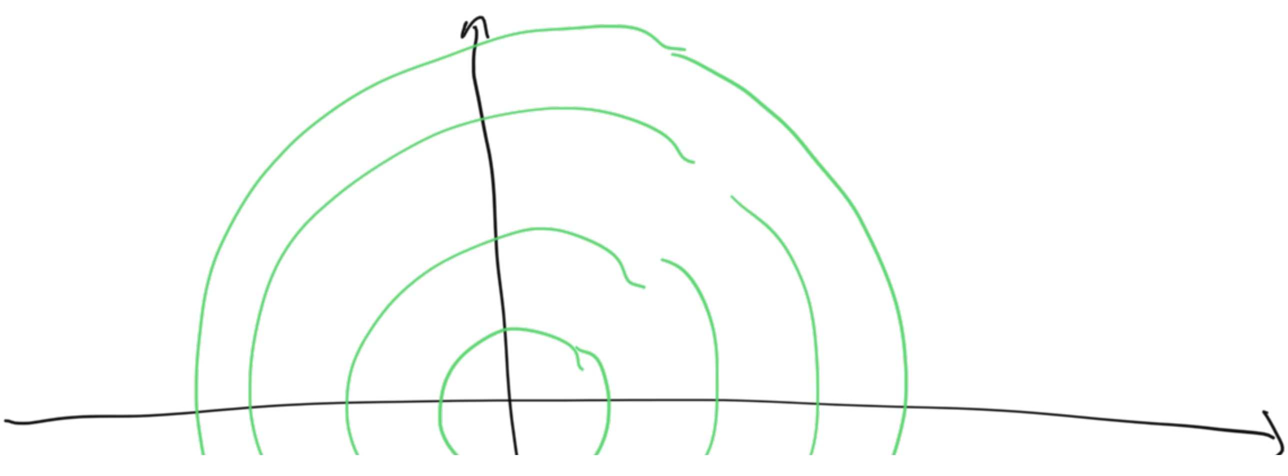
$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2i\begin{pmatrix} a \\ b \end{pmatrix}$$

$$2b = 2ia$$

$$w = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

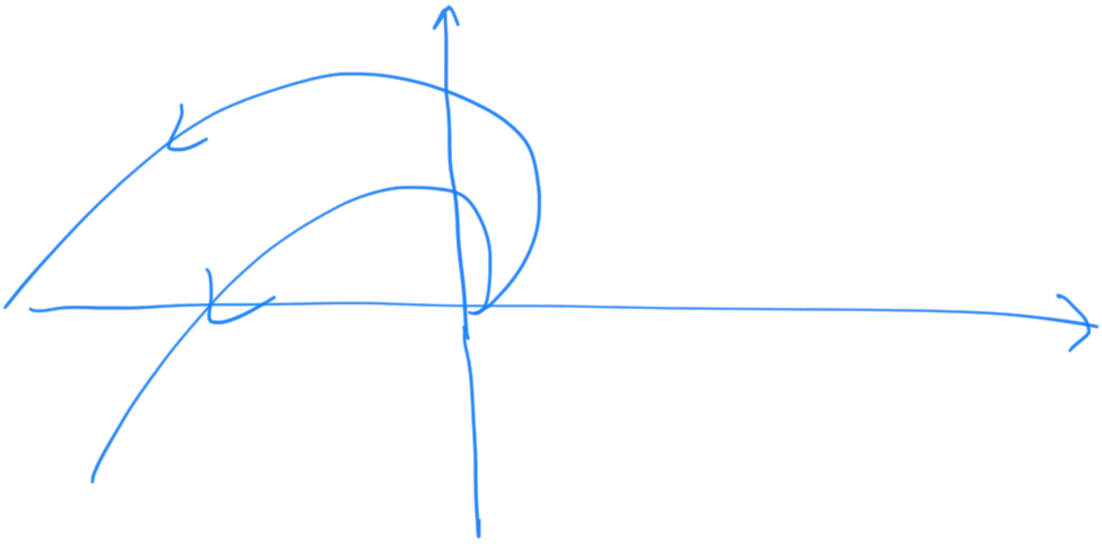
$$y(t) = C_1\left(\cos 2t\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$+ C_2\left(\sin 2t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$



center

$T > 0$



$T < 0$