The Phase Plane

Last time: Saw how to solve

$Y' = AY$ explicitly A 2×2

$Y(t) = (x(t), y(t))$

$$Y'(t) = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Easier to analyze $x(t)$ and $y(t)$ together instead of separately.

The plot of $(x(t), y(t))$ in the xy-plane is called the phase portrait of the system.

This technique will work well for <u>autonomous</u> <u>systems</u>, i.e. systems where RHS doesn't depend on t.

$$Y' = AY$$

- $\det(A) \neq 0$
- $\det(A) = 0$

will start with this

If $\det(A) \neq 0$, then only

equilibrium pt is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Recall, the behavior of solutions was determined by eigenvalues of A.

Case I: $\Delta > 0$

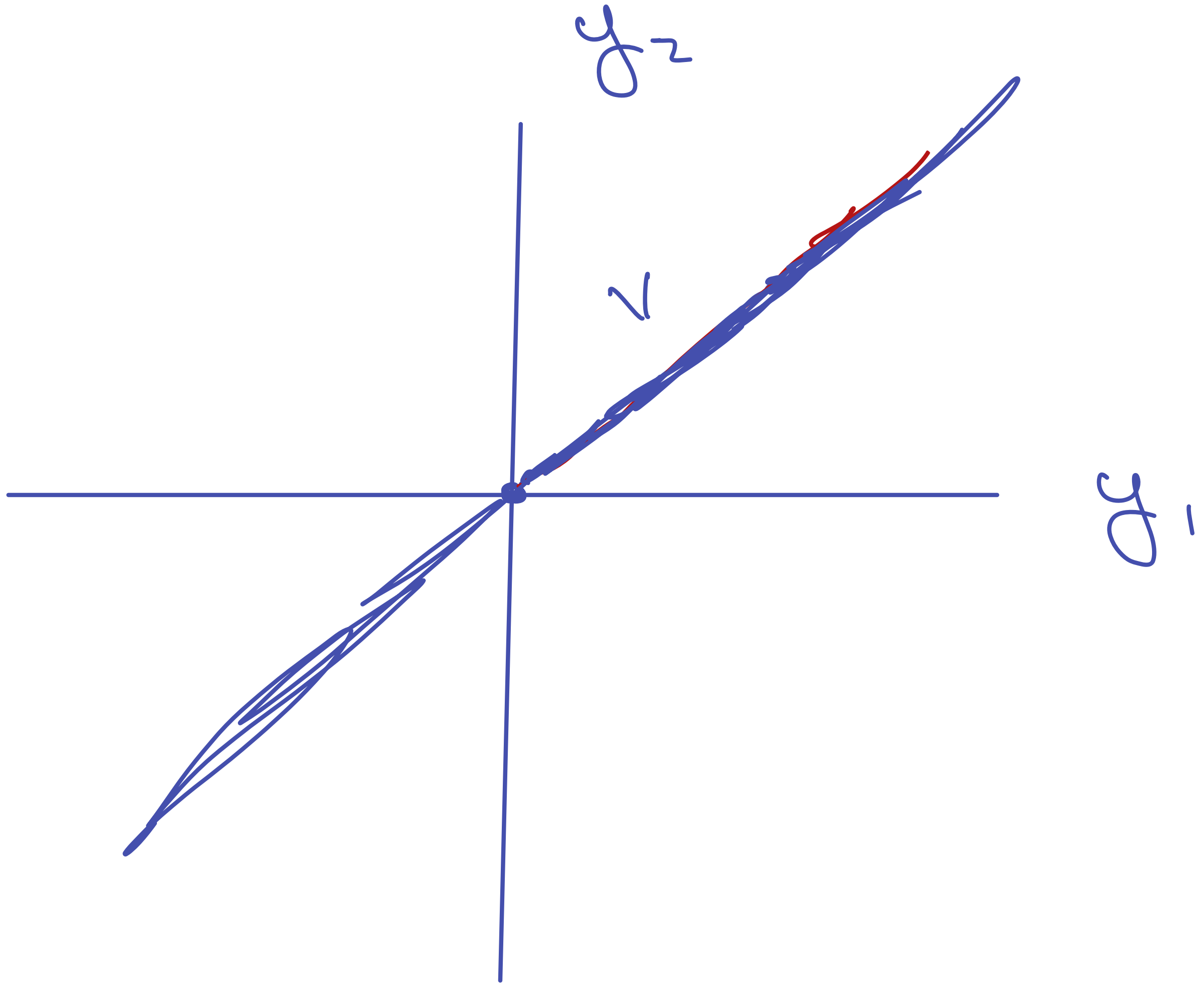
two real distinct eigenvalues λ_1, λ_2

$$Y(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$$

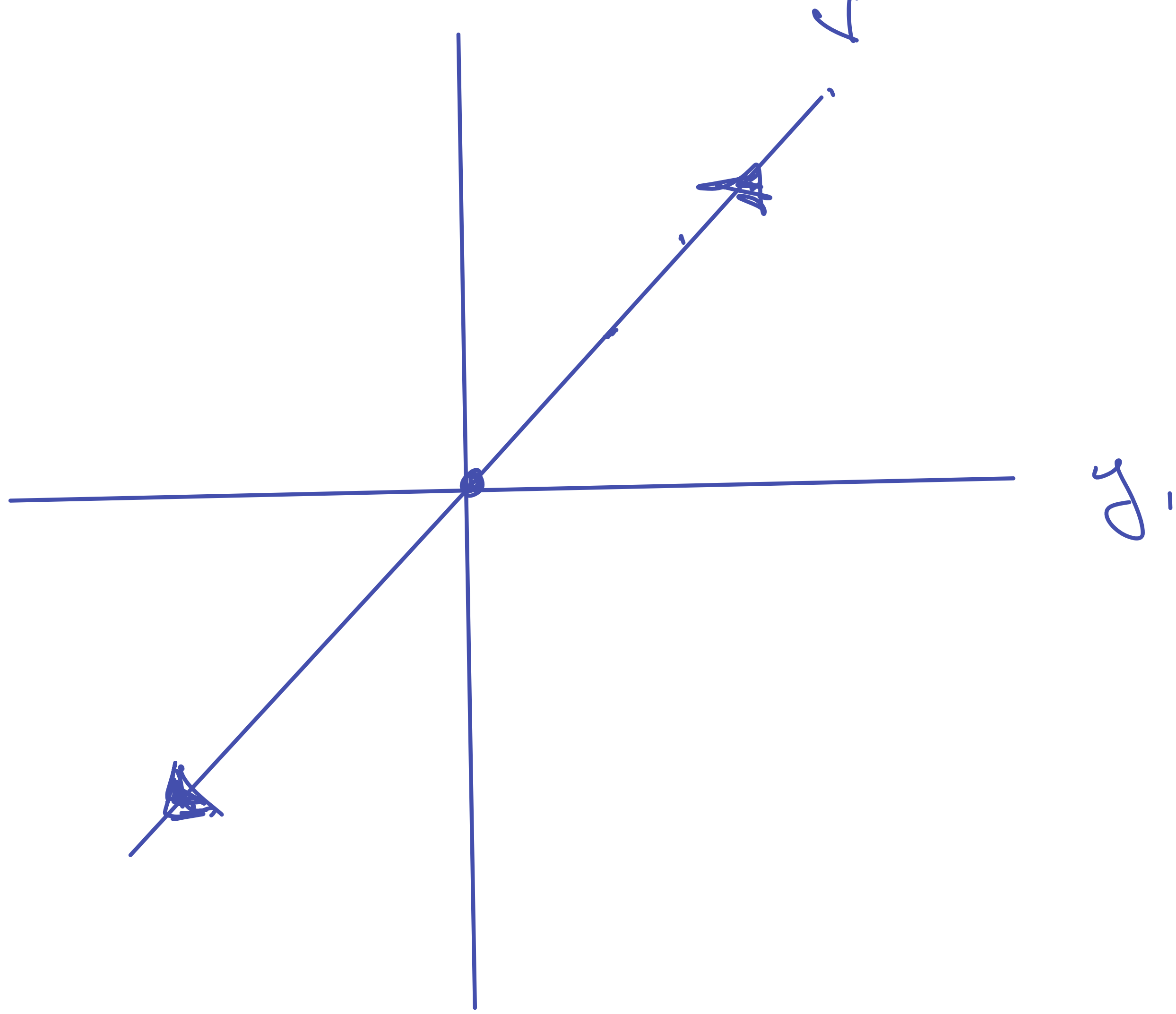
taking c_1 or $c_2 = 0$ recovers the special sol'ns

$$c_1 V_1 e^{\lambda_1 t} \quad \text{or} \quad c_2 V_2 e^{\lambda_2 t}$$

let's just consider $\underset{\sim}{c v} e^{\lambda t}$

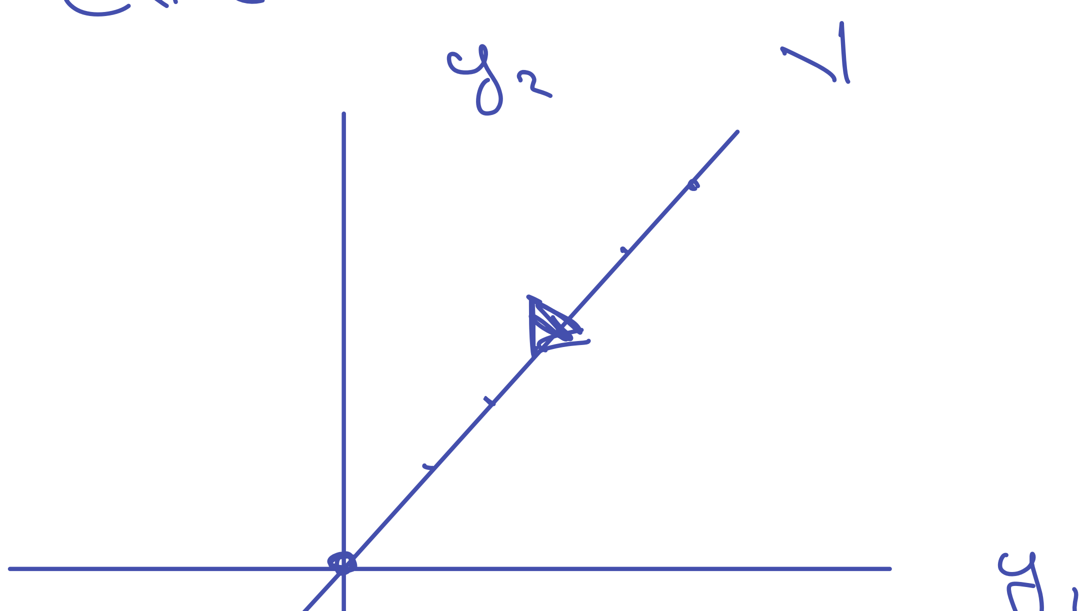


Depending on if $c > 0$ or $c < 0$, solutions go in direction of v or negative.

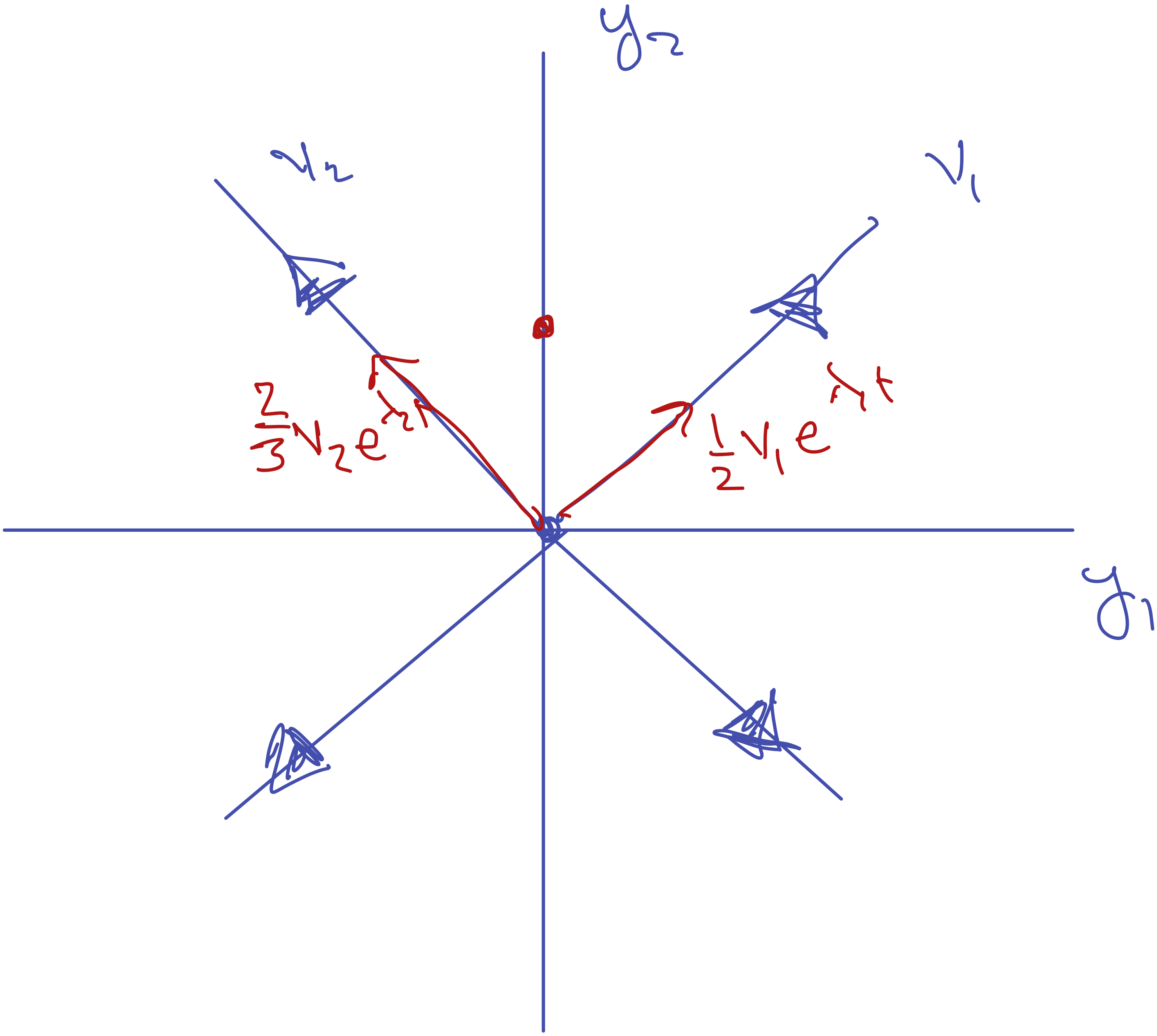


$\lambda > 0: \quad c v e^{\lambda t} \to \infty \quad \text{as } t \to \infty$

$\lambda < 0: \quad c v e^{\lambda t} \to 0 \quad \text{as } t \to \infty$

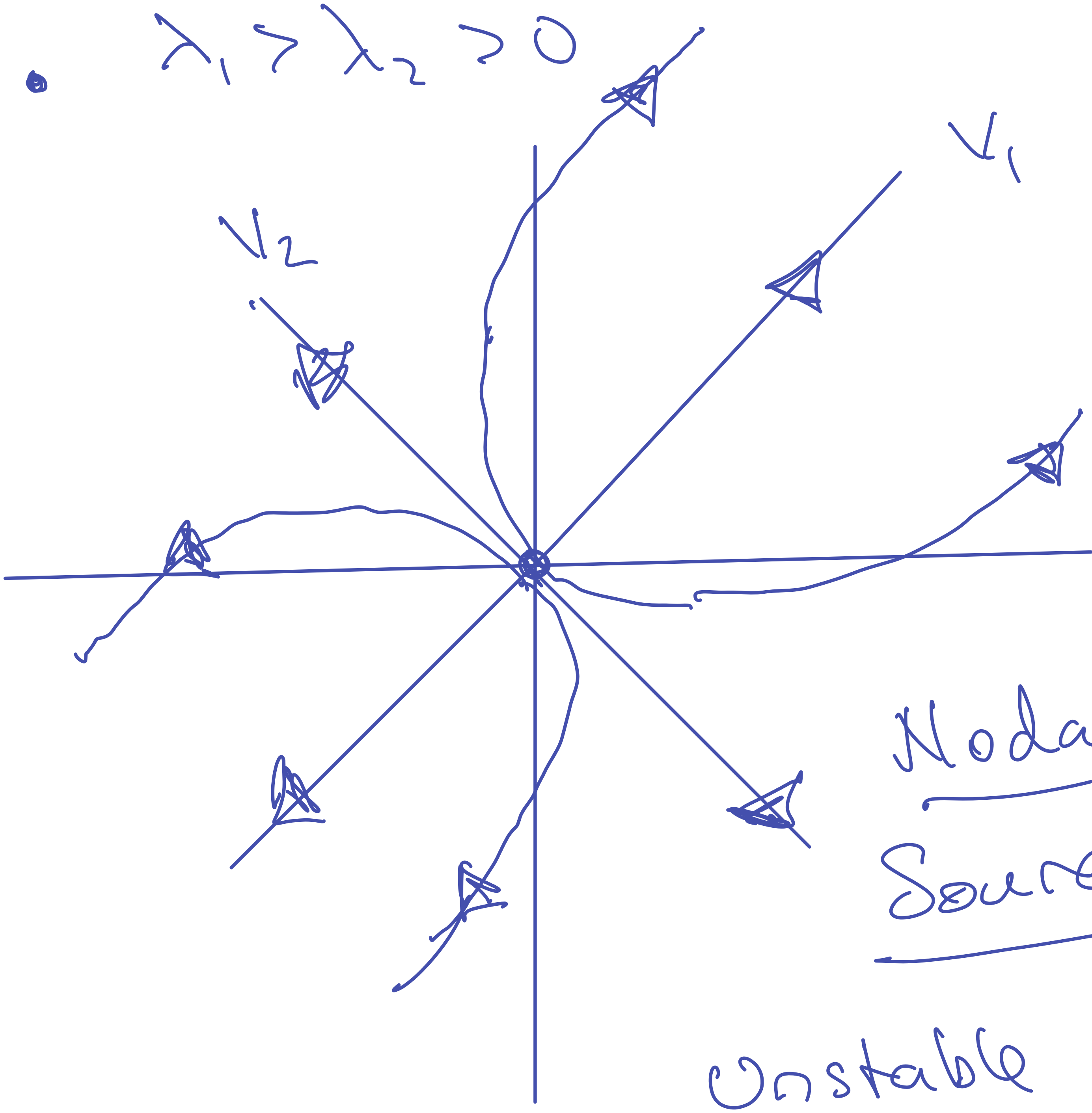


What about if $c_1, c_2 \neq 0$?



Depending on λ_1, λ_2, get different pictures:

- $\lambda_1 > \lambda_2 > 0$



Nodal Source

Unstable

What do trajectories look like?

$$c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$e^{\lambda_1 t}\left(c_1 v_1 + c_2 v_2 e^{(\lambda_2-\lambda_1)t}\right)$$

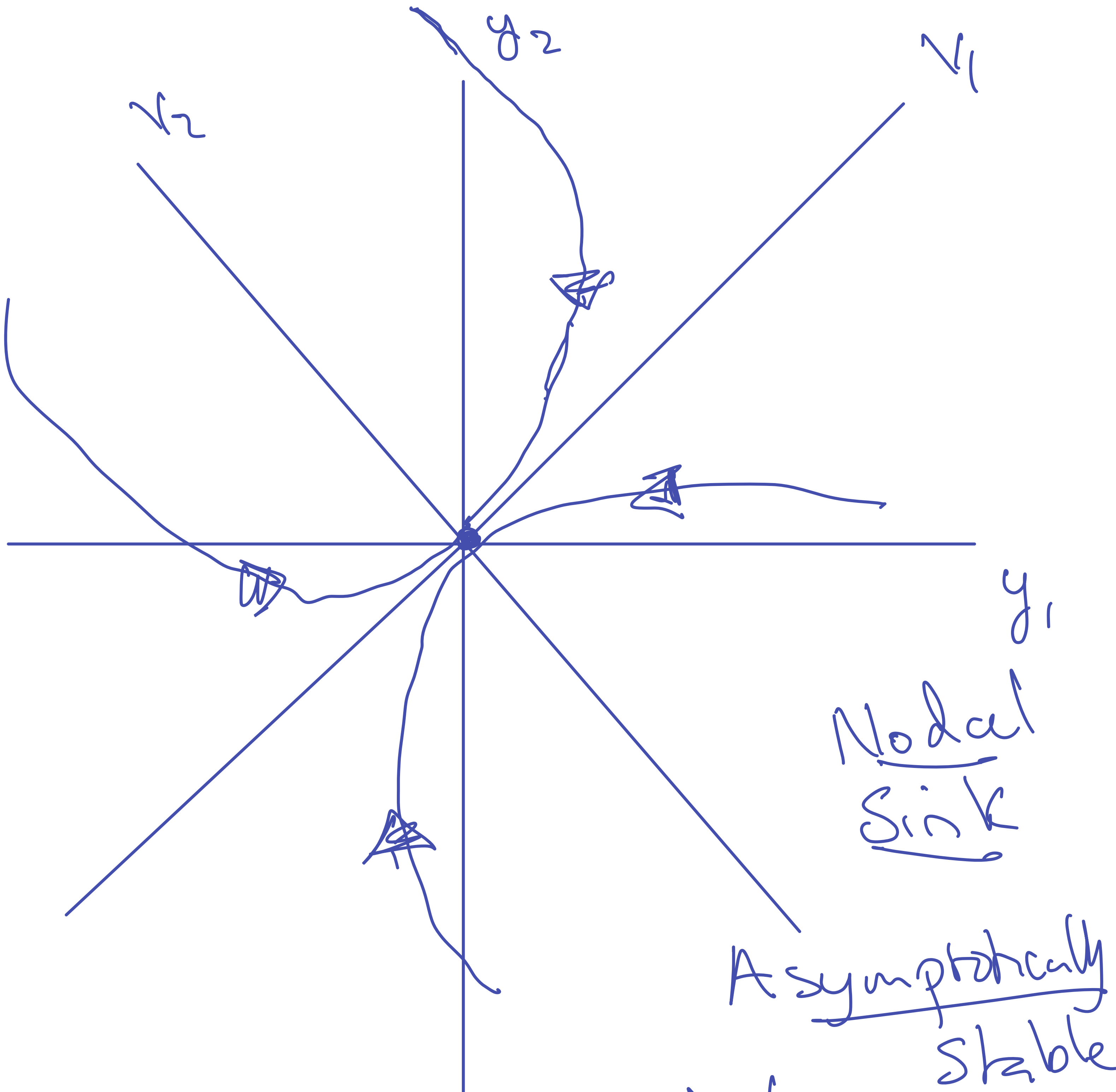
As $t \to \infty$, second term dies since $\lambda_2 - \lambda_1 < 0$

So sol^n $\simeq c_1 v_1 e^{\lambda_1 t}$

As $t \to -\infty$, $(\lambda_2 - \lambda_1)t > 0$

sol^n $\simeq c_2 v_2 e^{\lambda_2 t}$

- $0 > \lambda_1 > \lambda_2$



$$c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$e^{\lambda_1 t}\left(c_1 v_1 + c_2 v_2 e^{(\lambda_2-\lambda_1)t}\right)$$

As $t \to \infty$ $\quad \lambda_2 - \lambda_1 < 0$

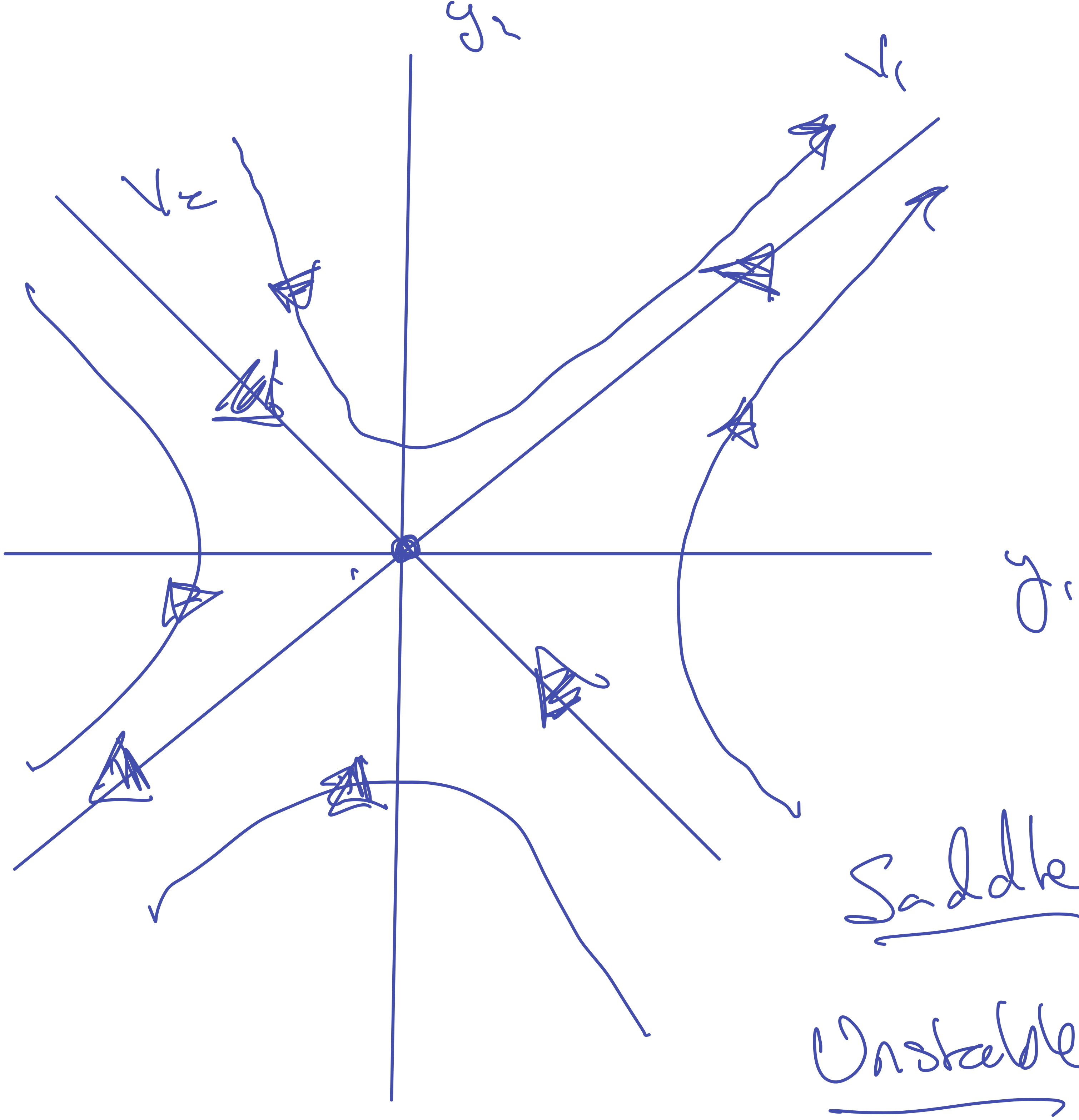
Sol^n $\approx$ $e^{\lambda_1 t} c_1 v_1$

As $t \to -\infty$

$$e^{\lambda_2 t}\left(c_1 e^{(\lambda_1-\lambda_2)t} + c_2 v_2\right)$$

Sol^n $\approx$ $e^{\lambda_2 t} v_2 c_2$

- $\lambda_1 > 0 > \lambda_2$



Saddle

Unstable

$$c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

$$\approx c_1 v_1 e^{\lambda_1 t} \quad \text{as } t \to \infty$$

$\sim c_2 v_2 e^{\lambda_2 t}$ as $t \to -\infty$

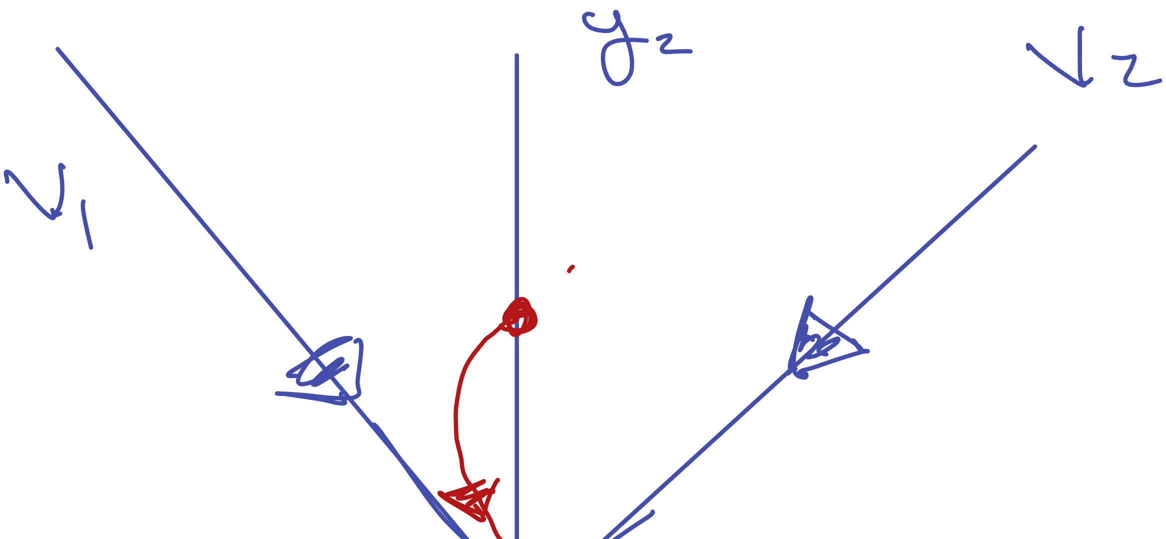
Ex: $Y' = AY$

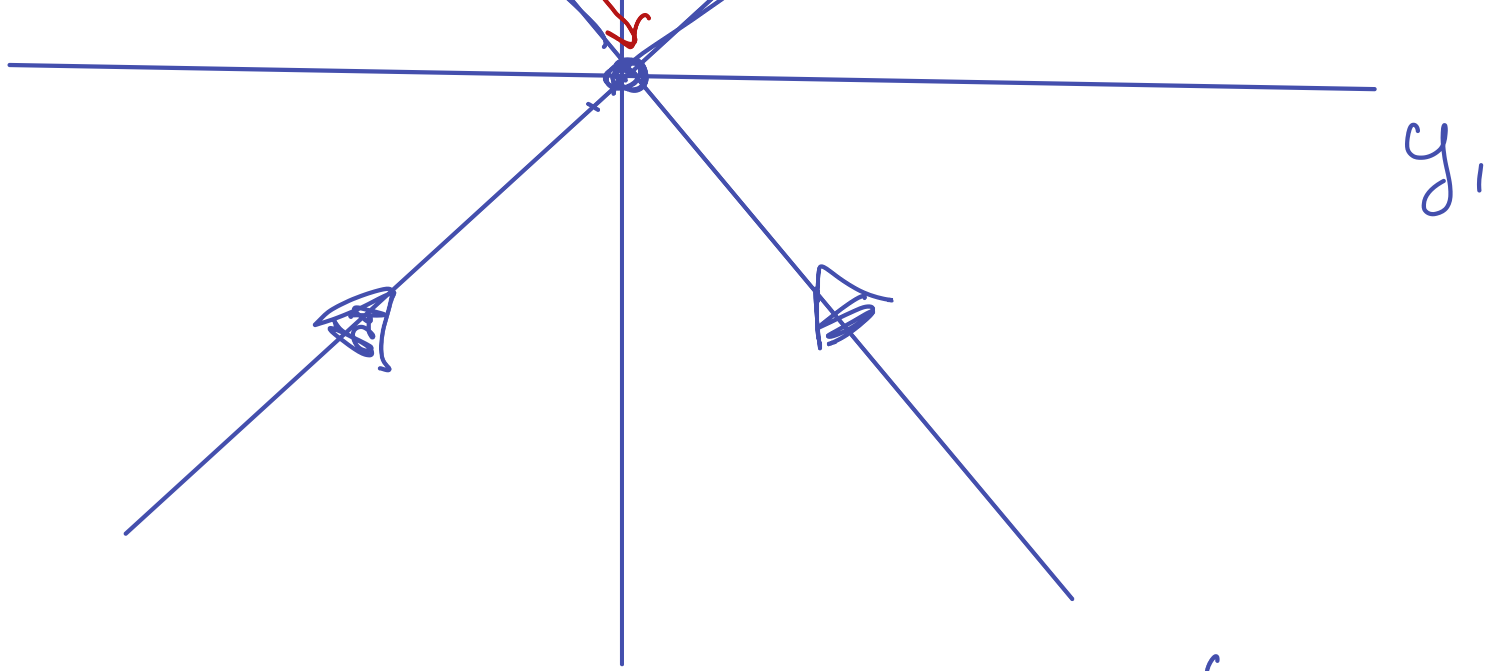
$$A = \begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix}$$

Eigenvalues: $\lambda_1 = -2$

$\lambda_2 = -4$

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$





Nodal Sink. What's trajectory of sol^n with

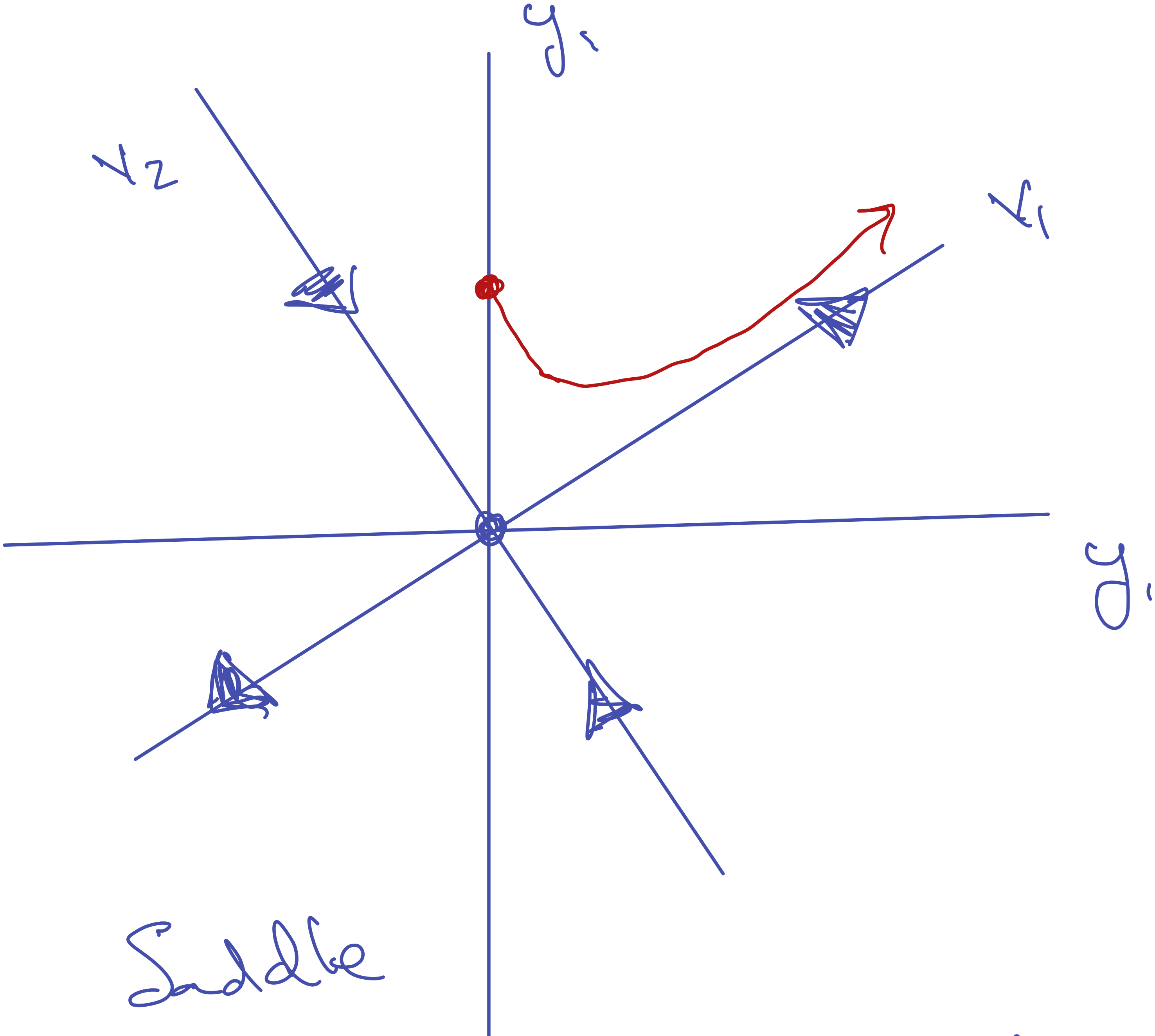
$$Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Ex: $\quad Y' = AY$

$$A = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$$

$\lambda_1 = 3$ $\quad v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = -3$ $\quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



Saddle

What's trajectory of sol^n

with $Y(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

Case II: $\Delta < 0$

$\lambda = a + bi \qquad \bar{\lambda} = a - bi$

Recall the general sol^n:

$$Y(t) = c_1 e^{at}\left(\cos(bt)v_1 - \sin(bt)v_2\right) + c_2 e^{at}\left(\sin(bt)v_1 + \cos(bt)v_2\right)$$

We don't have the nice half-line solutions in this case.

• $a = 0$

$$Y(t) = c_1\left(\cos(bt)v_1 - \sin(bt)v_2\right)$$
$$+ c_2\left(\sin(bt)v_1 + \cos(bt)v_2\right)$$

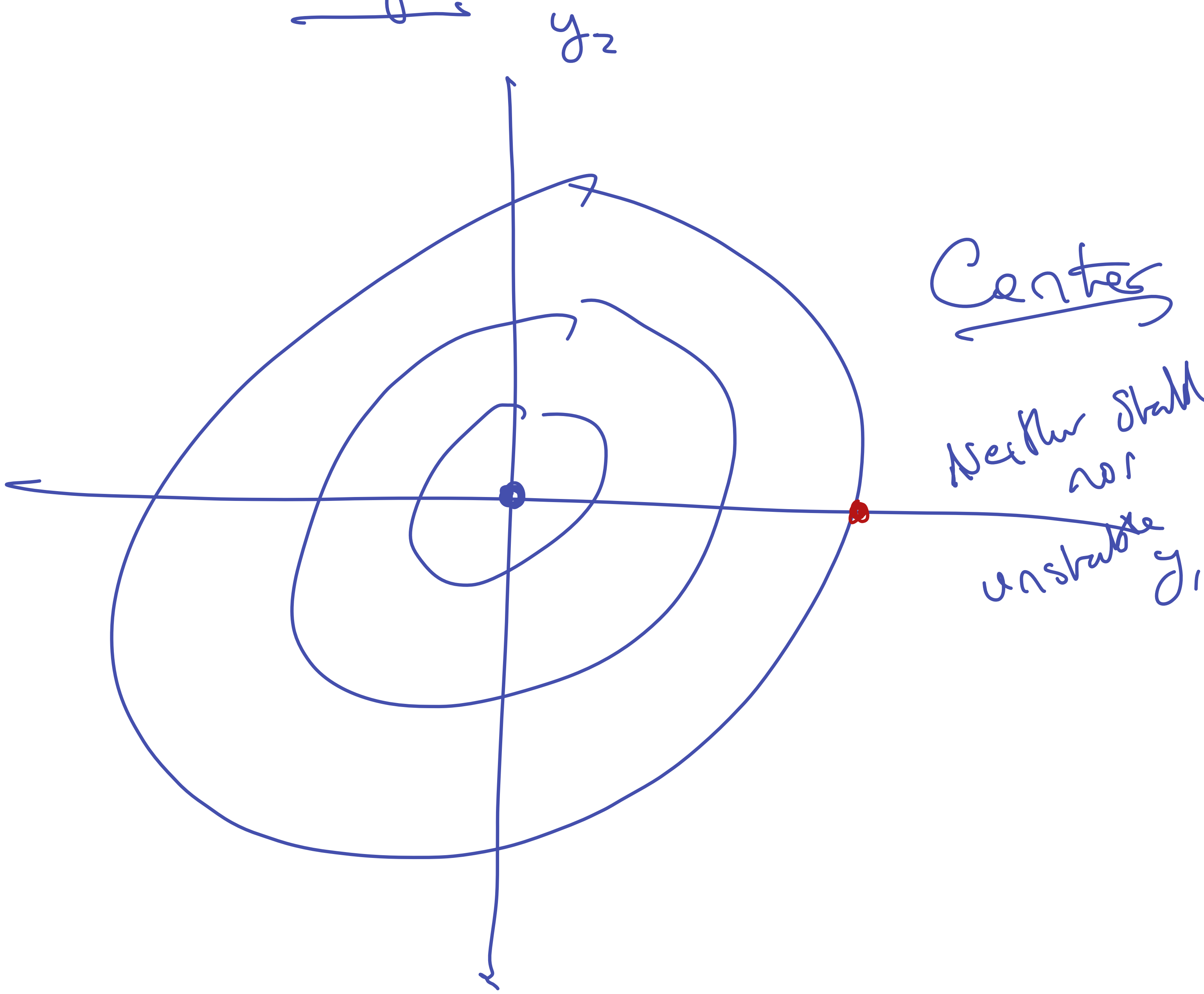
$\sin(bt)$ $\cos(bt)$ are periodic of period $2\pi/|b|$

$\Rightarrow$ $Y(t)$ is also periodic

$\Rightarrow$ $Y(t)$ is a closed curve in the y_1y_2-plane.

Using some linear algebra, one can show that the orbit of $Y(t)$ is

an ellipse.



Center

Neither stable nor unstable

to determine the direction

of rotation you can

just pick a test point

and see what Y' will

be.

This works b/c Y' is

tangent to trajectory.

Ex: $Y' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} Y$

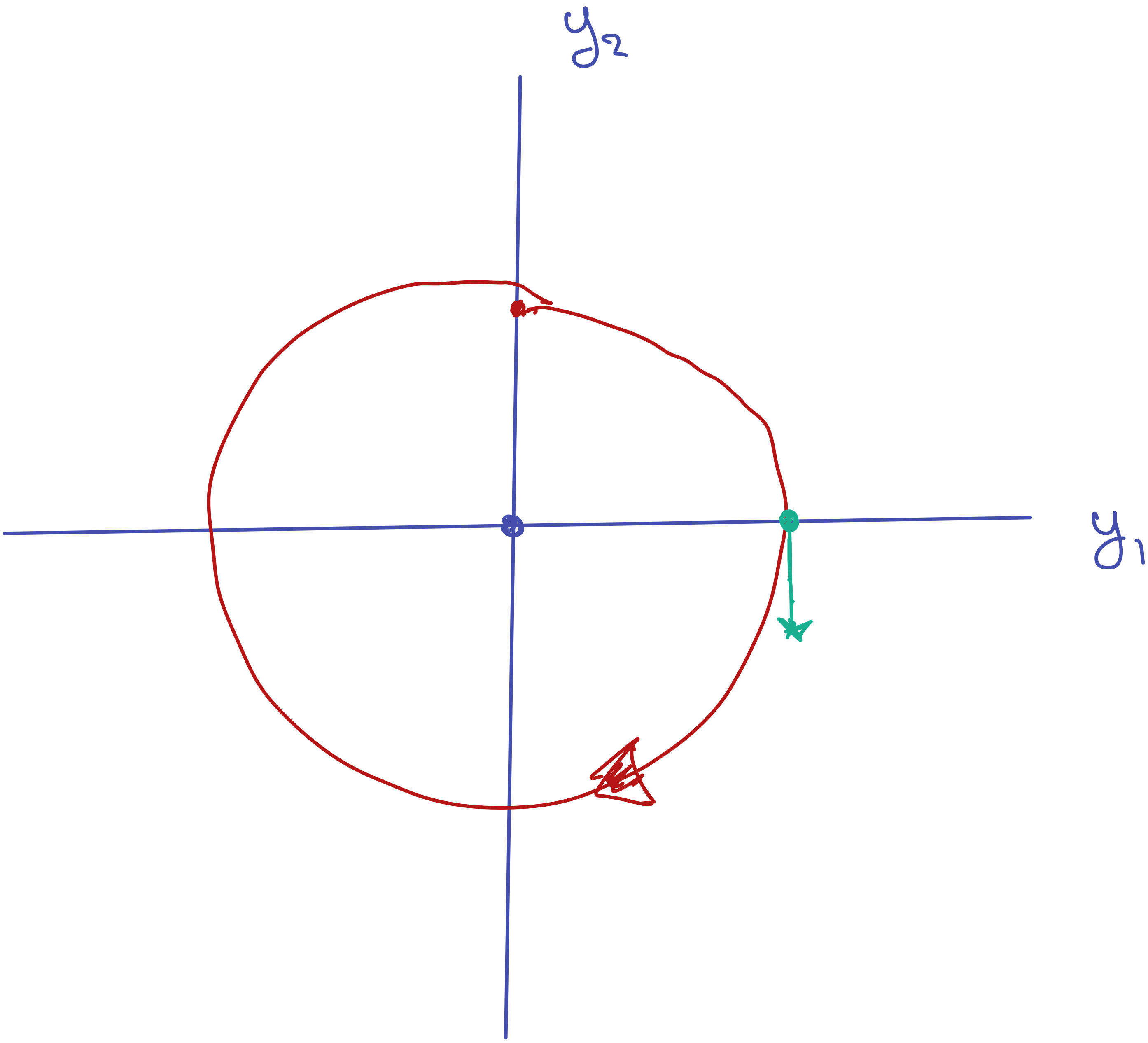
$\lambda = 2i \qquad v = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$e^{2it}\begin{pmatrix} 1 \\ i \end{pmatrix} \rightsquigarrow$$

$$\left(\cos(2t) + i\sin(2t)\right)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + i\begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}$$

$$Y(t) = C_1 \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + C_2 \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}$$



One can show that

$y_1^2 + y_2^2 = \text{constant}$ by

Showing that $(y_1^2 + y_2^2)' = 0$.

So trajectory will be a circle.

E.g. trajectory through $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

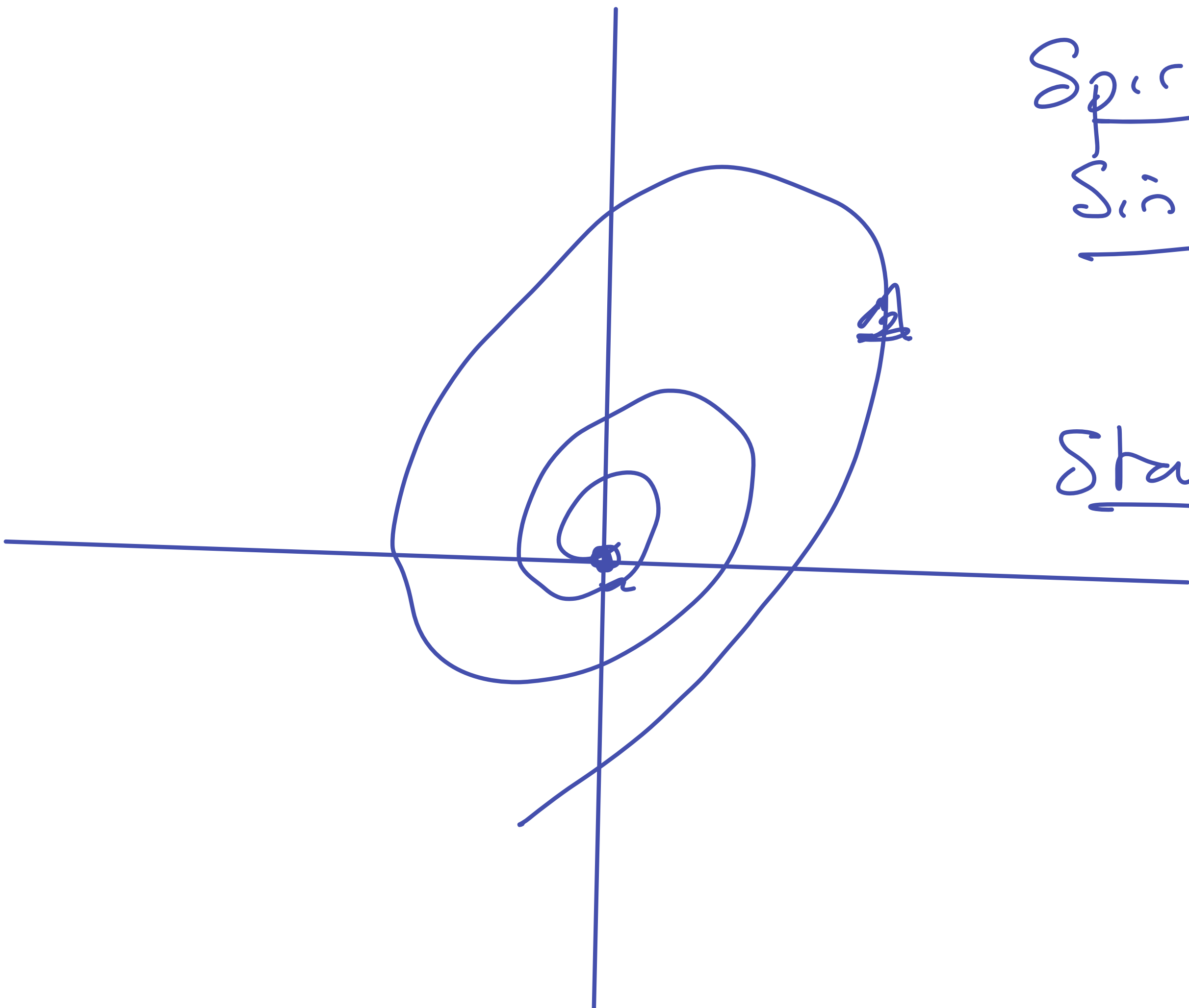
to determine rotation, need to pick a test point.

$$Y' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

$\Rightarrow$ cw rotation.

- $a > 0$ and $a < 0$ remain

<u>$a < 0$</u>



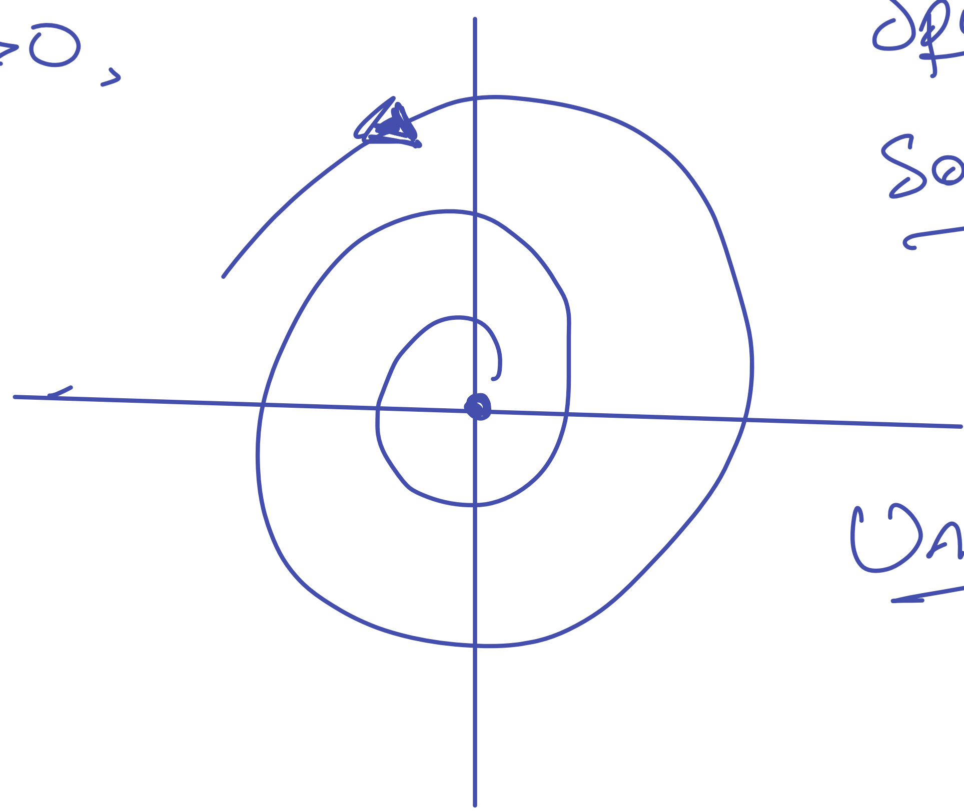
<u>Spiral</u>
<u>Sink</u>

<u>Stable</u>

the exponential term is decaying which causes us to spiral in towards origin along elliptical path.

Like before, can determine the direction of the spiraling by picking a test point.

if $a > 0$,

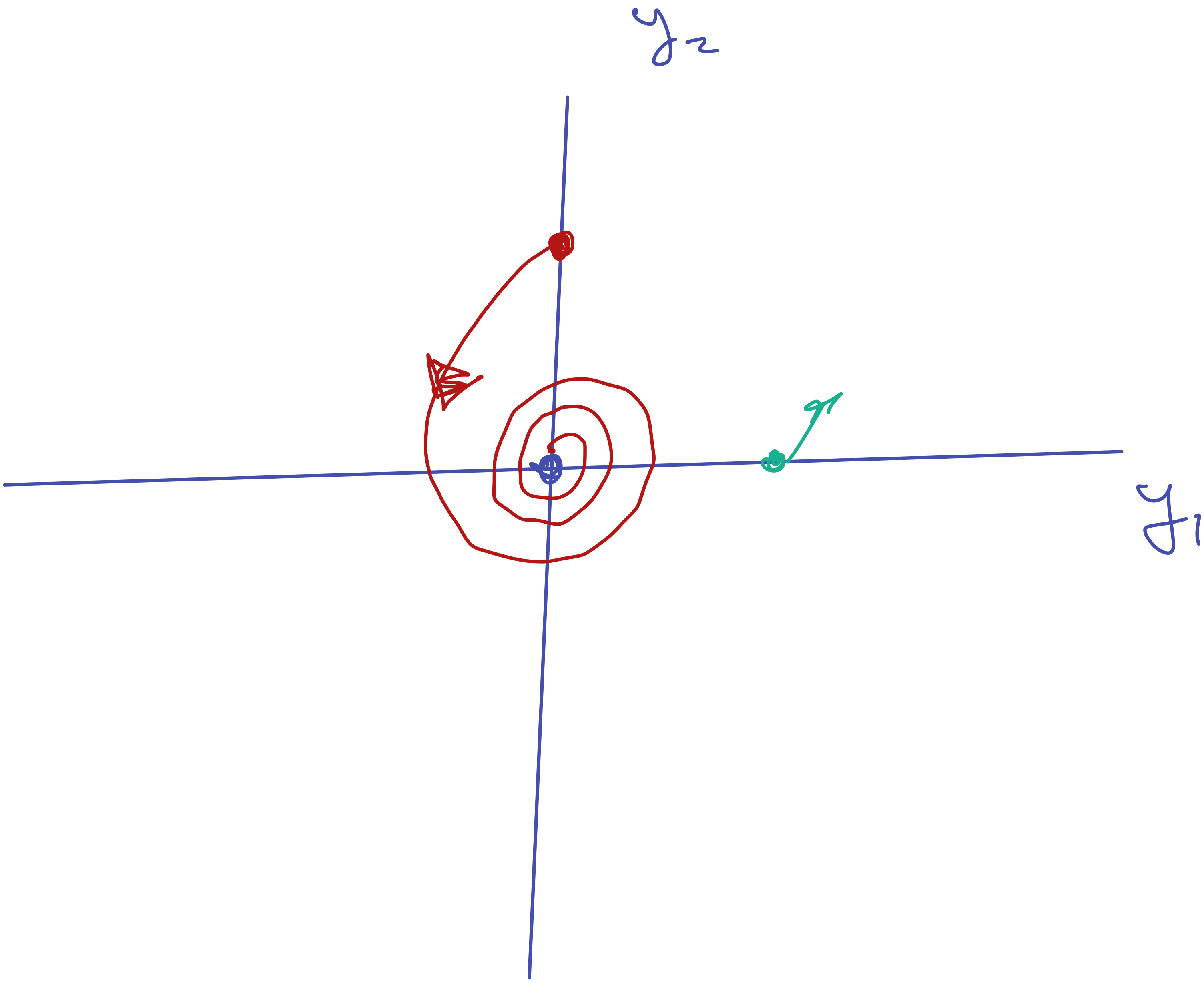


Spiral Source

Unstable

Ex: $Y' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix} Y$

$\lambda = -1 + 2i \qquad V = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$



What's trajectory going through $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

To determine rotation direction plug in test point:

$$y' = \begin{pmatrix} 1 & -4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

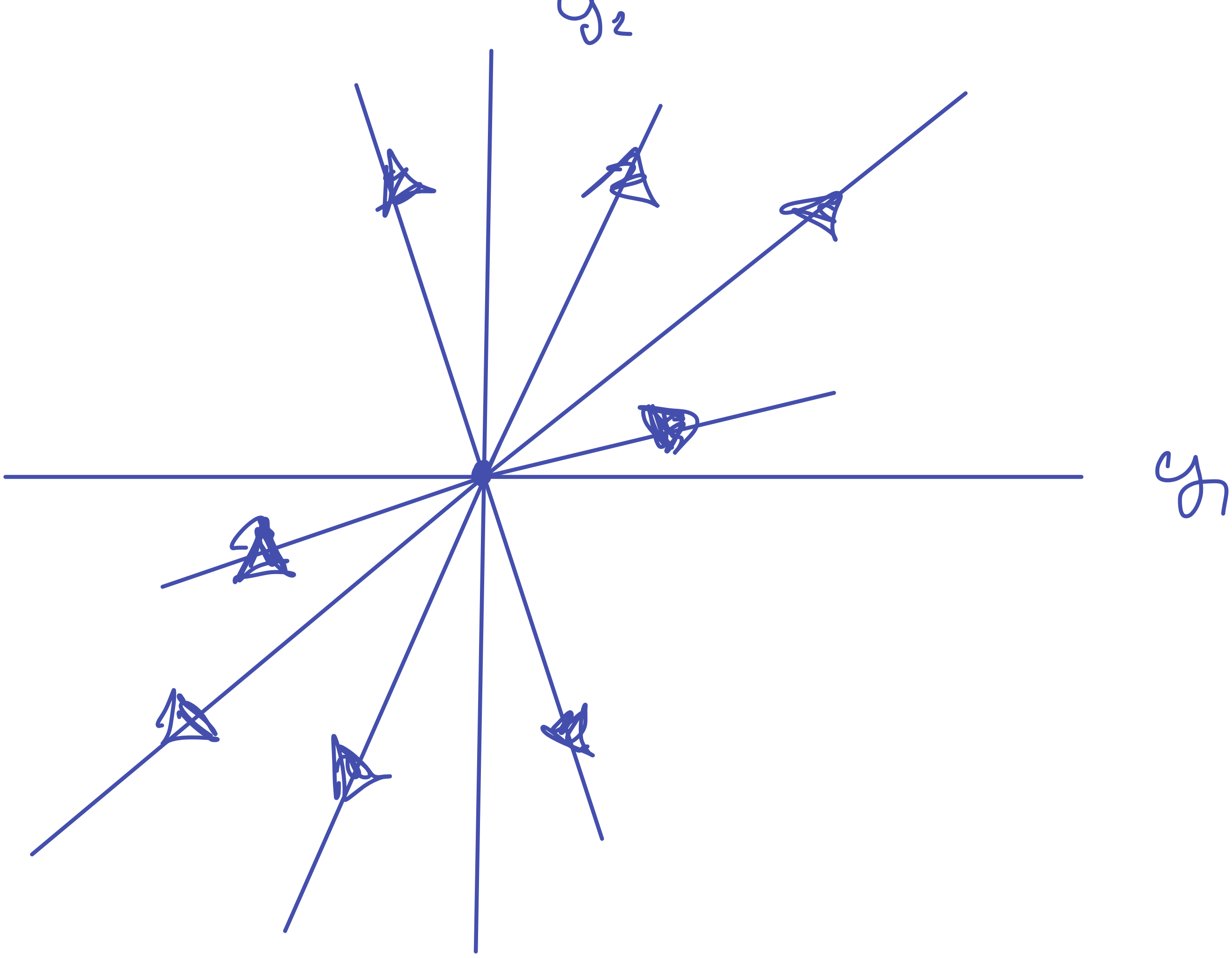
So ccw rotation.

Case III $\Delta = 0$

- two linearly independent eigenvectors

in this case, all v are eigenvectors

$Y(t) = ve^{\lambda t}$, v arbitrary



Same pictures as what we were doing earlier. (but happens for all v)

- Only one dimensional eigen space

$$Y(t) = c_1 v e^{\lambda t} + c_2 e^{\lambda t}(vt + v_2)$$

$$(A - \lambda I) v_2 = v$$

$$Y(t) = e^{\lambda t}(c_1 v + c_2 v t + c_2 v_2)$$

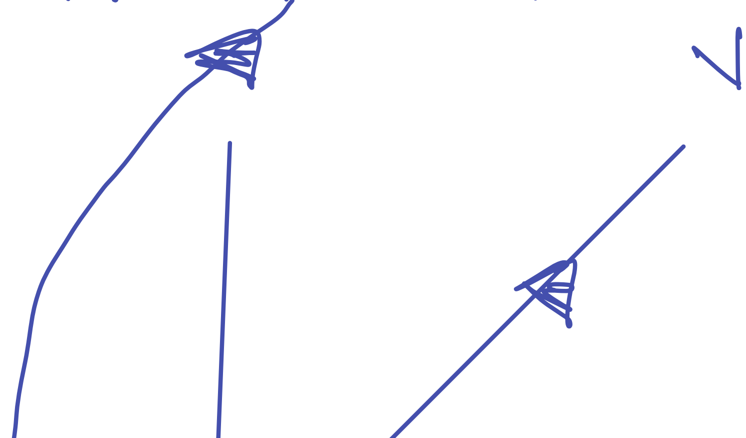
$$e^{\lambda t}((c_1 + c_2 t) v + c_2 v_2)$$

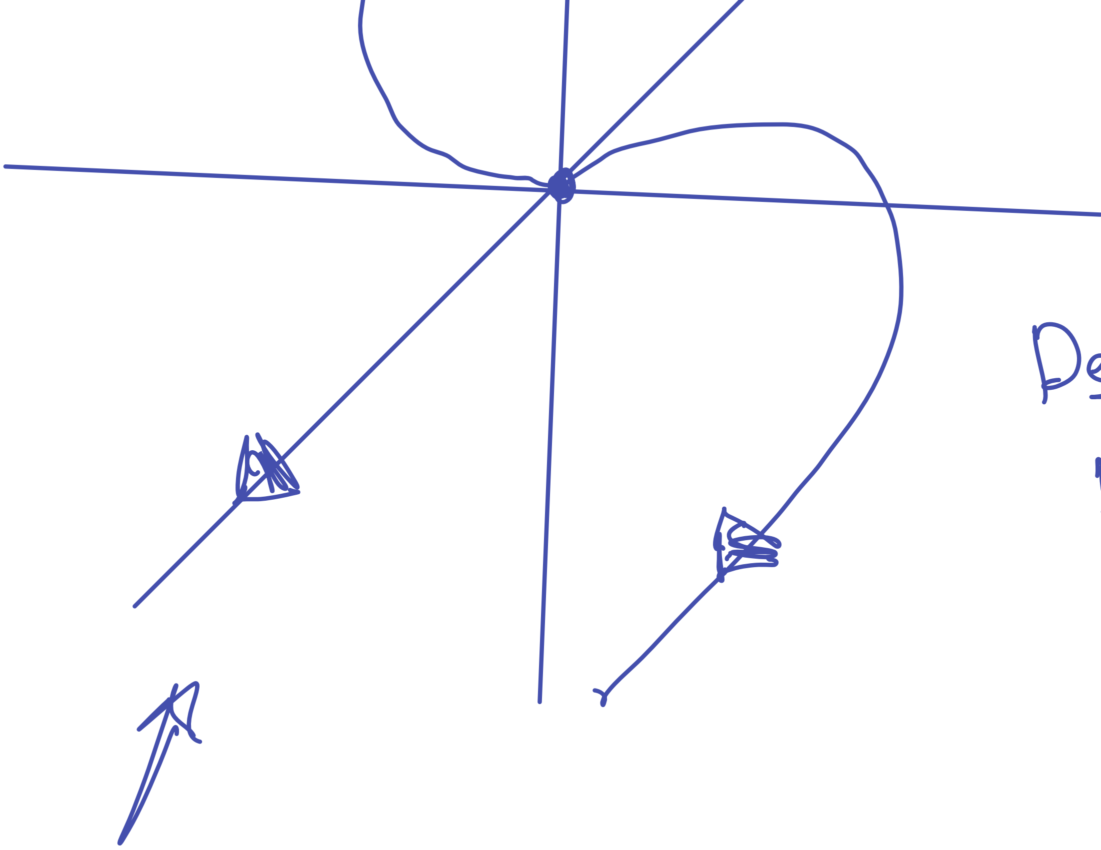
Half line sol^n: $c_1 e^{\lambda t} v$

As $t \to \infty$, $\approx e^{\lambda t}((c_1 + c_2 t) v)$

So move parallel to v

(either in or out)



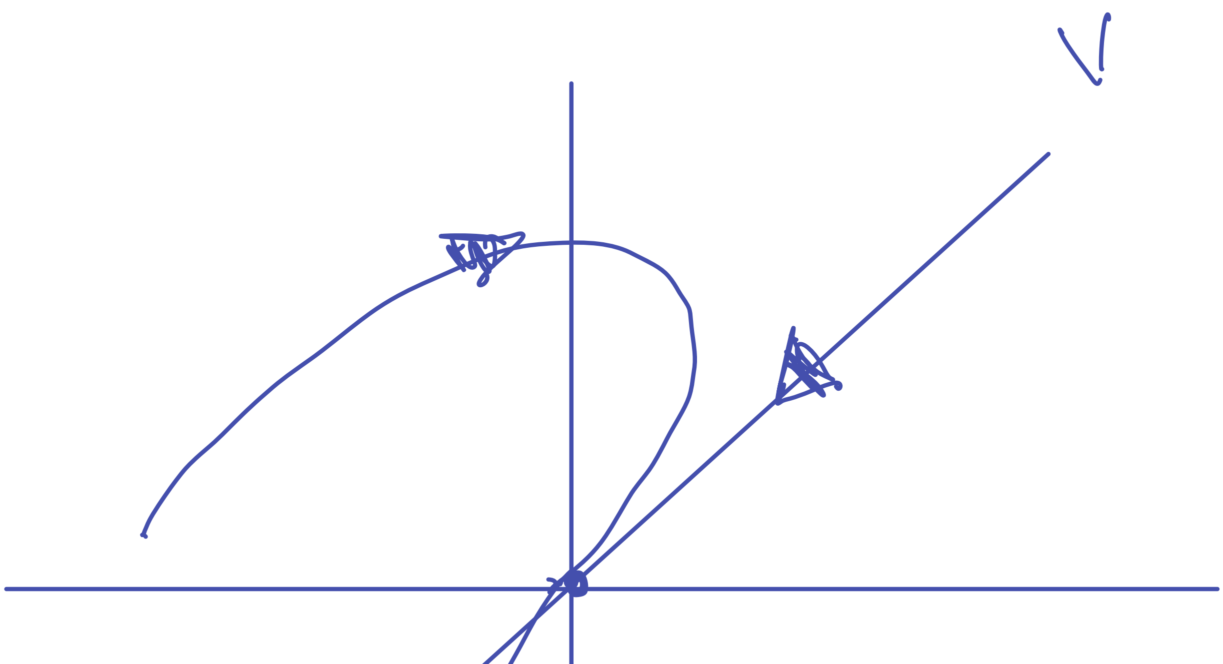


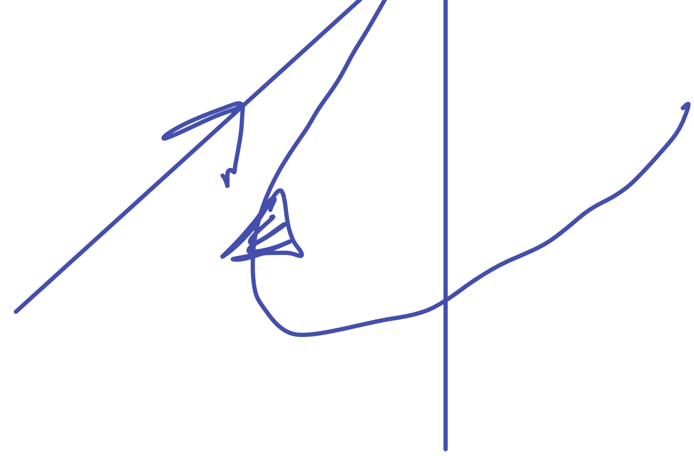
Degenerate Nodal Source

Unstable

$\lambda > 0$

rotation direction can be determined via a test pt.





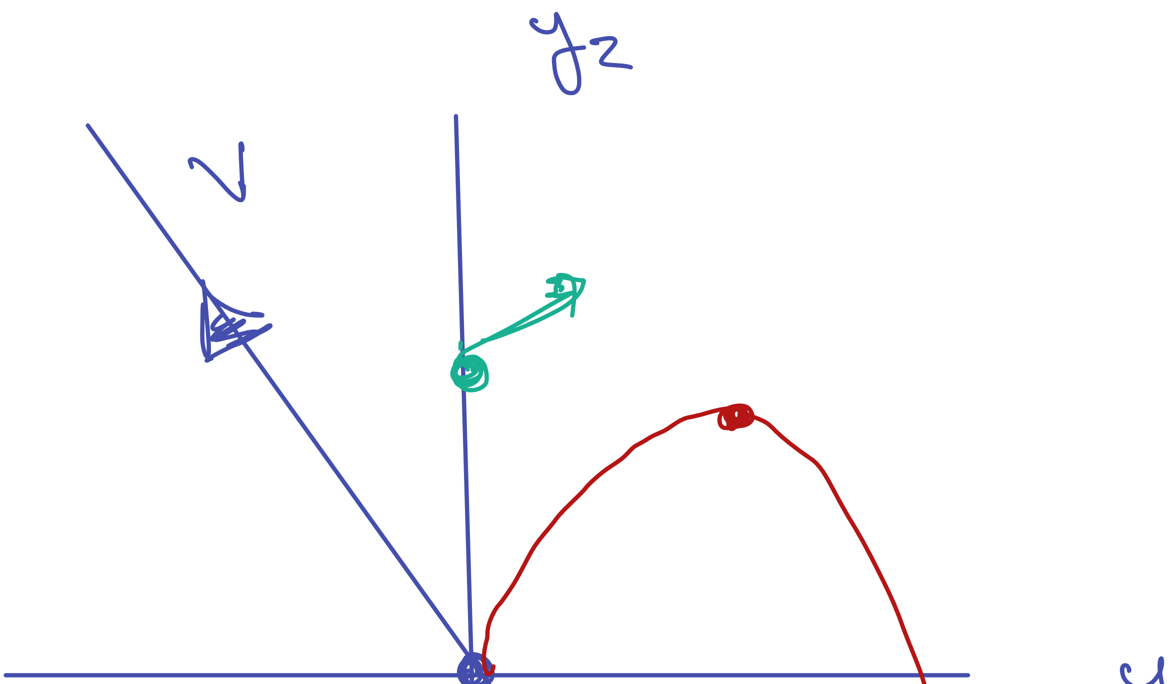
Degenerate Nodal Sink

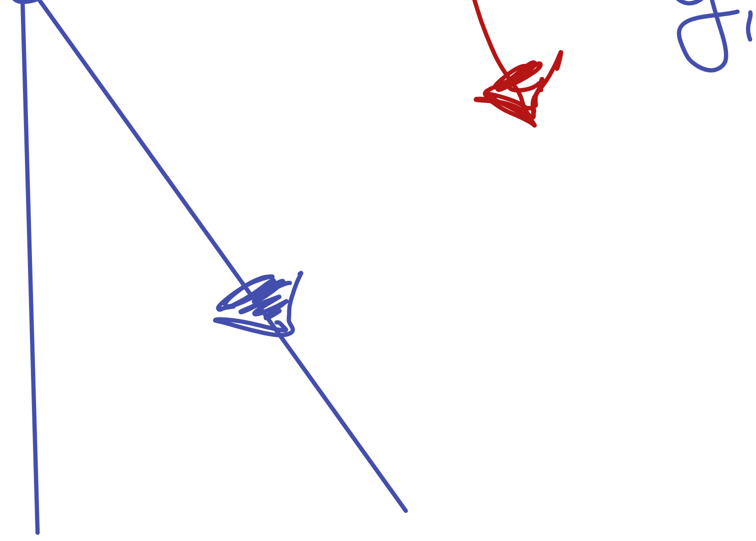
$\lambda < 0$

Asymptotically Stable

Ex: $Y' = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix} Y$

$\lambda = 5 \qquad v = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$





What's trajectory through $\begin{pmatrix}1\\1\end{pmatrix}$?

Have to check which way we spiral out.

Pick a test pt:

$$Y' = \begin{pmatrix}7 & 1\\ -4 & 3\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$$

Last case: $\underline{\det(A) = 0}$

in this case, one eigenvalue is 0.

$\lambda_1 = 0$ λ_2

V_1 V_2

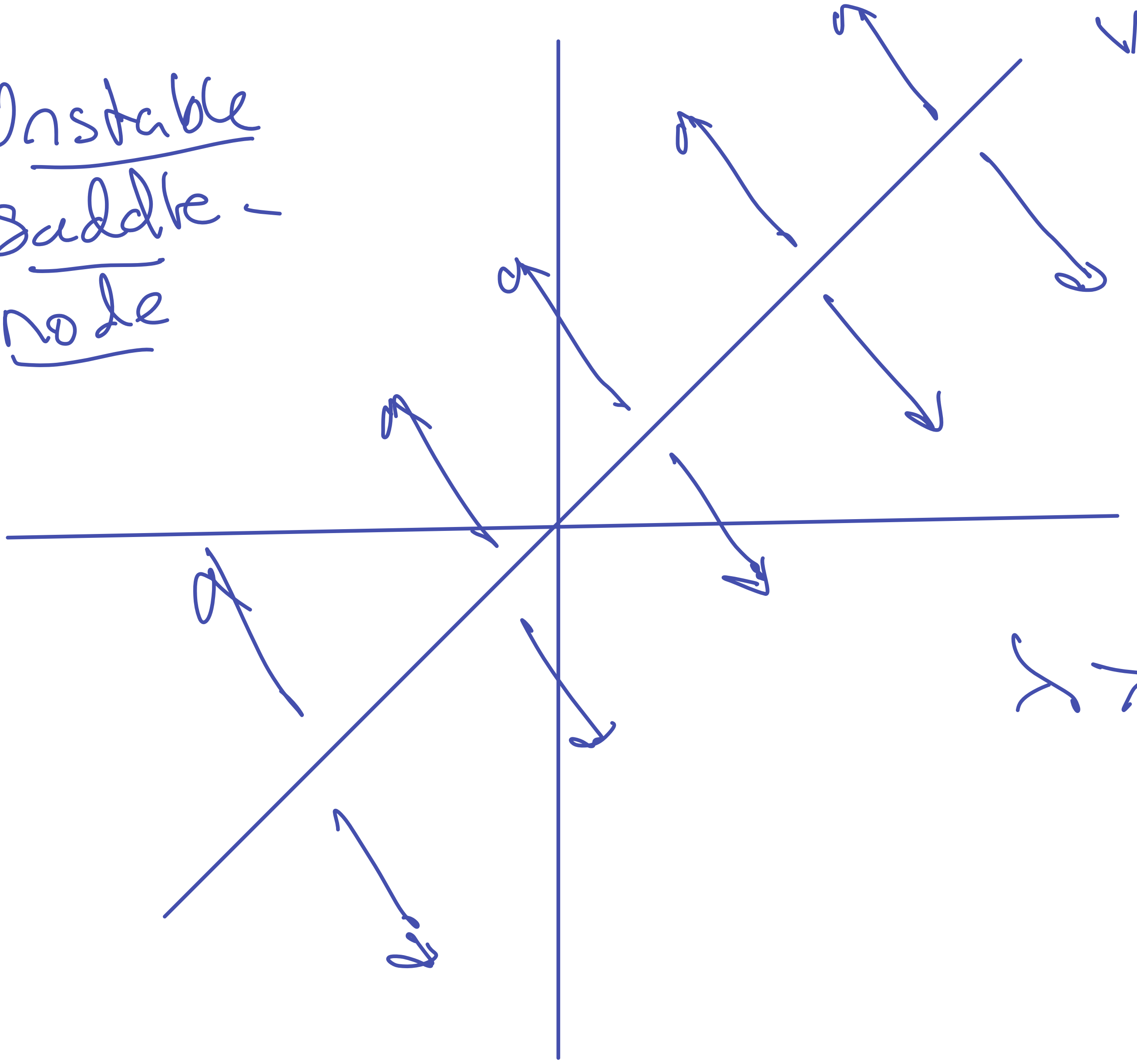
$$Y(t) = c_1 V_1 + c_2 V_2 e^{\lambda_2 t}$$

B/c V_1 has eigenvalue 0, all pts on V_1 are equilibrium pts

So we have an <u>equilibrium</u>

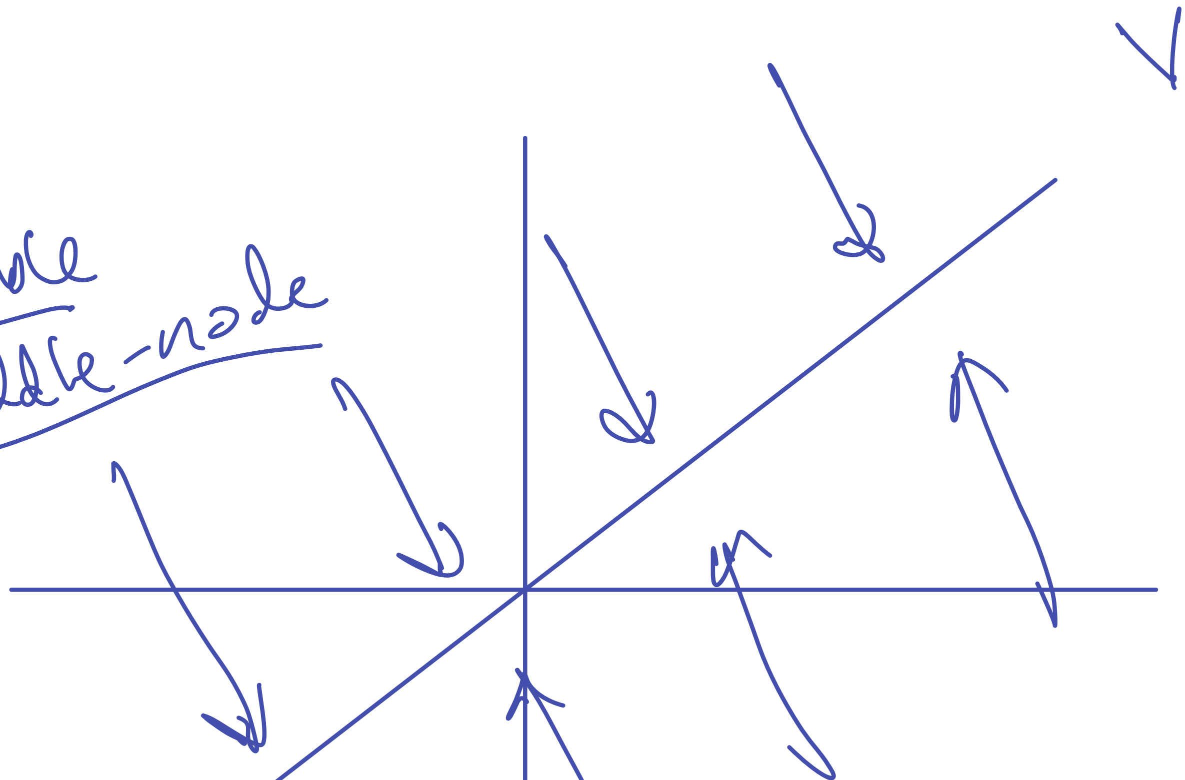
line -

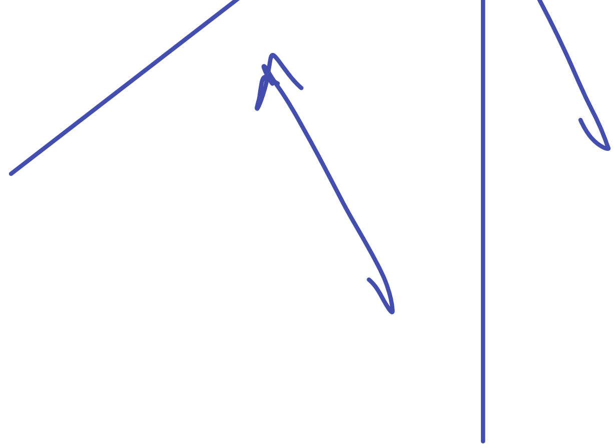
Unstable saddle-node



$\lambda > 0$

Stable saddle-node



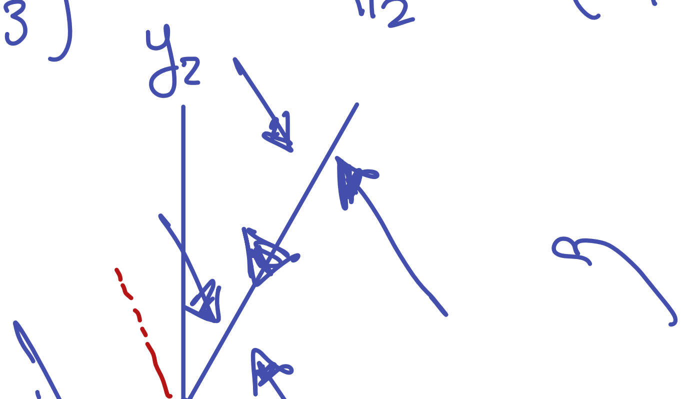


where the half line sol^n's are parallel to v_2.

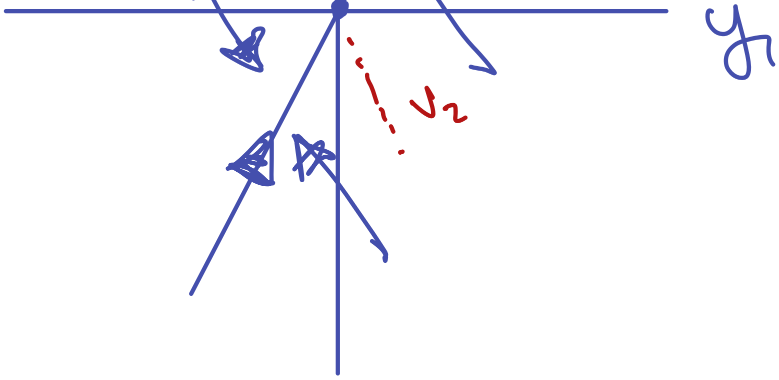
Ex: $Y' = \begin{pmatrix} -3 & 1 \\ 3 & -3 \end{pmatrix} Y$

$\lambda_1 = 0$ $\quad\quad$ $\lambda_2 = -4$

$v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\quad\quad$ $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



Phase



Portrait