Systems of ODEs

We will start with linear systems of ODEs w/ constant coeffs.

$x(t)$, $y(t)$

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

If we write $Y(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$,

then $Y'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$

$$Y' = AY \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

To solve these, we'll need

Some linear algebra.

Quick refresher of linear algebra:

$Ax = b$ $\rightsquigarrow$ System of equation

The matrix equation $Ax = b$ has a unique solution iff

$\det(A) \neq 0.$

For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \det(A) = ad - bc$

For larger matrices,

we can use cofactor expansion to compute det(A).

$$A = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{pmatrix}$$

A_{ij} $(i,j)^{th}$ minor — delete i^{th} row and j^{th} column.

$C_{ij} = (-1)^{i+j} \det(A_{ij})$ Cofactor

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for any } i.$$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

Let's cofactor expansion across first row.

$$1 \cdot \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \cdot \det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= 1 \cdot -3 - 2 \cdot (-6) + 3 \cdot (-3) = 0.$$

<u>Def</u>: The <u>trace</u> of A is defined as $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$

For any matrix A, the <u>characteristic polynomial</u> $p_A(x)$ is defined by

$$p_A(x) = (-1)^n \cdot \det(A - \lambda I) = \det(\lambda I - A)$$

An eigenvector of A is a non-zero vector v s.t. $Av = \lambda v$ for some $\lambda \in \mathbb{R}$.

We call λ the eigenvalue associated to v.

- Eigenvectors are directions where A acts by scaling by λ.

- If $Av = \lambda v$, then $(A - \lambda I)v = 0$

$\Longleftrightarrow$ $A - \lambda I$ not invertible $\Longleftrightarrow$

$\det(A - \lambda I) = 0$.

So eigenvalues are precisely the

roots of $p_A(x)$!

Ex: $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

$$p_A(x) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{pmatrix}$$

$$= -(4+\lambda)(5-\lambda) + 18 = -(20+\lambda-\lambda^2)+18$$

$$\lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1).$$

Eigenvalues: $-1, 2$.

$\lambda = -1$ $\quad Av = -v$

$$\begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{matrix} -4a + 6b = -a \\ -3a + 5b = -b \end{matrix}$$

$\Rightarrow \quad -3a + 6b = 0 \quad \Rightarrow \quad 2b = a.$

So any vector of the form
$c \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad c \in \mathbb{R}$ is an eigenvector.

$E_{-1} = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$\underline{\lambda = 2} \quad Av = 2v$

$$\begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix}$$

$-4a + 6b = 2a \quad \Rightarrow \quad -6a + 6b = 0$
$-3a + 5b = 2b$

$E_2 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$

$\Rightarrow \quad a = b.$

So $c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad c \in \mathbb{R}$ is an eigenvector.

$\{ v_1, \ldots, v_n \}.$

$\text{Span}\{ v_1, \ldots, v_n \} = \{ c_1 v_1 + \cdots + c_n v_n : c_i \in \mathbb{R} \}$

We say a set S is a spanning set of another set V if $V = \mathrm{Span}(S)$. If $V = \mathrm{Span}(S)$ and all vectors in S are linearly independent, we call S a <u>basis</u> for V.

(Recall: $\{v_1, \dots, v_n\}$ is linearly independent if $c_1 v_1 + \dots + c_n v_n = 0 \implies c_i = 0$ for all i).

E_λ = Eigenspace for λ

$= \{v : Av = \lambda v\}$.

We generally want to find bases of the E_λ.

Some facts about eigenvalues:

- $\mathrm{Tr}(A) = \sum_i \lambda_i$
- $\mathrm{Det}(A) = \prod \lambda_i$

In particular, for a 2×2 matrix, $p_A(x) = x^2 + ax + b$

$\Rightarrow \quad a = -\mathrm{tr}(A) \qquad b = \det(A)$

$$\boxed{p_A(x) = x^2 - \mathrm{tr}(A)x + \det(A)}$$

Ex: $A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}$

$$p_A(x) = x^2 - x - 2.$$

Let's go back to systems:
Start w/ 2×2.

$$y' = ay \rightsquigarrow y = Ce^{at}$$

might expect a similar story for $Y' = AY$:

$$Y(t) = ve^{At}$$ for some def. of e^{At}. This is true, but will see later.

Let hunt for solutions of the form

$y(t) = v e^{\lambda t}$ for some λ.

$y'(t) = \lambda v e^{\lambda t}$

$\iff \lambda v e^{\lambda t} = A(v e^{\lambda t})$

$\iff \lambda v = A v$

$\iff \lambda$ an eigenvalue of A!

$e^{\lambda t} v$ sol^n $\iff$ λ eigenvalue of A w/ eigenvector v.

So for the example above

$\lambda = -1, 2$ eigenvalues

$V_1 e^{-t}$ and $V_2 e^{2t}$ are

sol^n to $Y' = AY$.

Black Box theorems:

- $Y' = AY \qquad Y(0) = X_0$

  Still have existence/uniqueness thm.

- n linearly independent sol^n to a system of n ODEs

  $\Longrightarrow$ they form a basis for sol^n space.

- Sol^ns l.i. at some $t_0 \Longrightarrow$ linearly independent.

Reference: Chapter 8.

If A $n \times n$ matrix, $p_A(x)$ has n roots in $\mathbb{C}$ (w/ multiplicity).

Will have 3 cases like before:

- $\lambda \in \mathbb{R}$ distinct real roots
- $\lambda \in \mathbb{C} \setminus \mathbb{R}$ complex
- $\lambda \in \mathbb{R}$ repeated

The 2x2 Case

$$p_A(x) = x^2 - tr(A)\,x + Det(A)$$

$$\lambda = \frac{tr(A) \pm \sqrt{tr(A)^2 - Det(A)}}{2}$$

- $\Delta = tr(A)^2 - 4Det(A) > 0$
- $\Delta < 0$
- $\Delta = 0$

Linear Algebra fact: if λ_1 and λ_2 are distinct, and v_1, v_2 are associated eigenvectors, then v_1 and v_2 are linearly independent.

Why? $c_1 v_1 + c_2 v_2 = 0$. Appy A to both sides $\Longrightarrow$

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0.$$

Now, $c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 = 0$

$$\Longrightarrow \quad c_2 (\lambda_1 - \lambda_2) v_2 = 0.$$

$\Longrightarrow$ $c_2 = 0$ b/c $\lambda_1 \neq \lambda_2$.

$\Longrightarrow$ $c_1 = 0$ as well.

Therefore, if λ_1, λ_2 are distinct then $e^{\lambda_1 t} v_1$ and $e^{\lambda_2 t} v_2$ are linearly independent, b/c at $t=0$ we get v_1, v_2 which are l.i. $\Longrightarrow$

$e^{\lambda_1 t} v_1$ and $e^{\lambda_2 t} v_2$ l.i.

$$Y(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

for $c_1, c_2 \in \mathbb{R}$.

Ex: $Y' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} Y$

$$p_A(x) = x^2 - 3x - 4 = (x-4)(x+1)$$

$\Rightarrow$ Eigenvalues are $-1, 4$.

$\lambda = -1$

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} a \\ b \end{pmatrix}$$

$$a + 2b = -a \Rightarrow a = -b$$
$$3a + 2b = -b$$

$$E_{-1} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$\underline{\lambda = 4}$ $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 4 \begin{pmatrix} a \\ b \end{pmatrix}$

$$\begin{aligned} a + 2b &= 4a \\ 3a + 2b &= 4b \end{aligned} \Rightarrow \begin{aligned} 2b &= 3a \\ a &= 2/3\, b \end{aligned}$$

$E_4 = \text{Span}\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$Y(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{4t}$$

$$= \begin{bmatrix} -c_1 e^{-t} + 2c_2 e^{4t} \\ c_1 e^{-t} + 3c_2 e^{4t} \end{bmatrix}$$

$\underline{\Delta < 0}$

$\lambda = a+bi$, then $\bar{\lambda} = a-bi$.

$$Av = \lambda v \implies A\bar{v} = \bar{\lambda}\bar{v}$$

So if v is an eigenvector for λ, then $\bar{v}$ an eigenvector for $\bar{\lambda}$.

$ve^{\lambda t}$ and $\bar{v}e^{\bar{\lambda} t}$ are both sol^n to $Y' = AY$.

these are then linearly ind. by previous reasoning.

$$Y(t) = C_1 v e^{\lambda t} + C_2 \bar{v} e^{\bar{\lambda} t}$$

Prop: Z sol^n to $Z' = AZ$

$Z = X + iy$ $\quad x, y$ real valed vectors

- $\bar{Z}$ is a sol^n, so are x, y.

- If $Z, \bar{Z}$ l.i. $\Longrightarrow$ x, y linearly independent.

Proof: We have

$$x = \frac{1}{2}(Z + \bar{Z})$$

$$y = \frac{1}{2i}(Z - \bar{Z}).$$

In particular,

$$v = v_1 + i v_2$$

$$e^{\lambda t} v = \left[e^{at}(\cos(bt) + i\sin(t))\right](v_1 + i v_2)$$

$$= e^{at}(\cos(bt)v_1 - \sin(bt)v_2)$$

$$+ \; i \cdot e^{at}(\cos(bt)v_2 + \sin(bt)v_1)$$

$$Y(t) = c_1 e^{at}(\cos(bt)v_1 - \sin(bt)v_2)$$

$$+ \; c_2 e^{at}(\cos(bt)v_2 + \sin(bt)v_1)$$

Ex: $A = \begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix}$

$$p_A(x) = x^2 - 2x + 2$$

$$\lambda = \frac{2 \pm \sqrt{-4}}{2} \to 1 \pm i.$$

$\underline{\lambda = 1+i}$

$$\begin{pmatrix} 0 & 1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (1+i) \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \quad b = (1+i)a$$

$$-2a + 2b = (1+i)b$$

$$E_{1+i} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \right\}$$

$$v = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$Y(t) = c_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(1+i)t} + c_2 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(1-i)t}$$

If you want, you can

write this in terms of

Sin/cos.

$$Y(t) = C_1 e^{t} \begin{bmatrix} \cos(t) \\ \cos(t) - \sin(t) \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \sin(t) \\ \sin(t) + \cos(t) \end{bmatrix}$$

<u>$\Delta = 0$</u>

- We have 2 eigenvectors: If l.i. V_1, V_2

$e^{\lambda t} V_1$, $e^{\lambda t} V_2$ are sol's

and are l.i. b/c l.i. when plugging in 0.

$$y(t) = C_1 V_1 e^{\lambda t} + C_2 V_2 e^{\lambda t}$$

Since E_λ has 2 l.i. eigenvectors

$$E_\lambda = \mathbb{R}^2$$

$\Rightarrow$ <u>any</u> vector is eigenvector.

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$y(t) = \begin{pmatrix} c_1 e^{\lambda t} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 e^{\lambda t} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t} \qquad c_1, c_2 \in \mathbb{R}.$$

If $E_\lambda = \mathrm{Span}\{v\}$:

then we only have

$v e^{\lambda t}$ as an obvious

solution. Let's try the

usual trick:

look for solution of the form $wte^{\lambda t}$ for some w.

$$Y' = AY$$

$$we^{\lambda t} + \lambda wte^{\lambda t} = Awte^{\lambda t}$$

$$w + \lambda wt = Awt$$

But plug in $t=0$:

$$w = 0,$$

uh oh!

New guess:

$$Y(t) = V_1 te^{\lambda t} + V_2 e^{\lambda t}$$

for some v_1, v_2.

$y' = Ay$:

$$v_1 e^{\lambda t} + \lambda t v_1 e^{\lambda t} + \lambda v_2 e^{\lambda t} = Av_1 t e^{\lambda t} + Av_2 e^{\lambda t}$$

$$v_1 + \lambda t v_1 + \lambda v_2 = Av_1 t + Av_2$$

Plug in $t=0$:

$$v_1 + \lambda v_2 = Av_2$$

$$\Rightarrow \boxed{v_1 = (A - \lambda I) v_2}$$

Plug this back in:

$$v_1 + (\lambda I - A) v_2 = (A - \lambda I) t v_1$$

has to hold for all t

$$\Rightarrow (A - \lambda I) v_1 = 0.$$

$$Av_1 = \lambda v_1$$

$\Rightarrow$ $\boxed{v_1 \text{ eigenvector}}$

$v_1 = v$ Choose v_2 s.t.

$$v = (A - \lambda I) v_2$$

$$Y(t) = c_1 t v e^{\lambda t} + c_1 v_2 e^{\lambda t}$$

this is l.i. from $ve^{\lambda t}$

$\Rightarrow$ general sol^n is

$$\boxed{Y(t) = c_1 v e^{\lambda t} + c_2 (vt + v_2) e^{\lambda t}}$$

Ex: $A = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}$

$$p_A(x) = x^2 - 10x + 25$$
$$= (x-5)^2$$

$$\begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 5\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{aligned} 7a + b &= 5a \\ -4a + 3b &= 5b \end{aligned} \implies b = -2a$$

$$E_5 = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

take $v = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Need

$$(A - 5I)v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \quad 2a+b = 1$$
$$-4-2y = -2$$
$$\Rightarrow \quad b = 1-2a$$

take $a=0, \quad b=1.$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$y(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t} + c_2 \left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) e^{5t}$$

$$= \begin{pmatrix} c_1 e^{5t} + c_2 t e^{5t} \\ -2c_1 e^{5t} - 2c_2 t e^{5t} + c_2 e^{5t} \end{pmatrix}.$$