Math 33B : 7/14

# System of linear differential equations (with constant coefficients)

$$y' + p(t)\,y + q(t) = 0$$

$$y'' + p y' + q y = r$$

$$y_1' = a_{11} y_1 + a_{12} y_2 + \dots + a_{1n} y_n$$
$$y_2' = a_{21} y_1 + a_{22} y_2 + \dots + a_{2n} y_n$$
$$\vdots$$
$$y_n' = a_{n1} y_1 + a_{n2} y_2 + \dots a_{nn} y_n$$

$$y' = A y \qquad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

planar system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

## Strategy

$$p(\lambda) = \det(A - \lambda I)$$

$$A v_\lambda = \lambda v_\lambda$$

2 distinct real solutions $e^{\lambda_1 t} v$ $e^{\lambda_2 t} v$

2 distinct real solutions $e^{\lambda_1 t} v_1$ $e^{\lambda_2 t} v_2$

2 distinct complex solutions

$\lambda = a + bi \qquad \bar{\lambda} = a - bi$

$v = u + iw \qquad \bar{v} = u - iw$

$$e^{\lambda t} v \ , \quad e^{\bar{\lambda} t} \bar{v}$$

$$\begin{aligned} e^{\lambda t} v &= e^{(a+bi)t}(u + iw) \\ &= (e^{at}\cos bt + i e^{at} \sin bt)(u + iw) \\ &= (e^{at}\cos bt) u - (e^{at} \sin bt) w \\ &\quad + i (e^{at}\cos bt) w + i (e^{at} \sin bt) u \end{aligned}$$

$$(e^{at}\cos bt) u - (e^{at} \sin bt) w$$

$$(e^{at}\cos bt) w + (e^{at} \sin bt) u$$

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1 repeated real solution

easy $\quad A = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \lambda y_1 \\ \lambda y_2 \end{pmatrix}$$

$$e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix}, \quad e^{\lambda t}\begin{pmatrix}0\\1\end{pmatrix}$$

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difficult

$$e^{\lambda t} v$$

$$(A-\lambda I)w = v$$

$$e^{\lambda t} w + t e^{\lambda t} v$$

Why?

$$y' = Ay$$

$$A = PDP^{-1} \qquad D = \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$$

$$y' = PDP^{-1}y$$

$$(P^{-1}y)' = DP^{-1}y$$

$$z = P^{-1}y \quad \rightsquigarrow \quad z' = Dz$$

$$\begin{pmatrix}z_1'\\z_2'\end{pmatrix} = \begin{pmatrix}\lambda_1 z_1\\ \lambda_2 z_2\end{pmatrix}$$

$$y = Pz$$

$$\begin{pmatrix}\lambda & 1\\ & \lambda\end{pmatrix}$$

Example 1

$$x' = Ax, \quad A = \begin{bmatrix}-1 & 6\\-3 & 8\end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= \det\left(\begin{bmatrix} -1 & 6 \\ -3 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} -1-\lambda & 6 \\ -3 & 8-\lambda \end{bmatrix}\right)$$

$$= (-1-\lambda)(8-\lambda) - 6(-3)$$

$$= \lambda^2 - 7\lambda + 10$$

$$p(\lambda) = \lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 2)(\lambda - 5) = 0$$

$$\lambda = 2 \quad \text{or} \quad \lambda = 5$$

2 distinct real solutions

$$A v_1 = 2 v_1$$

$$A \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -x + 6y \\ -3x + 8y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\begin{cases} -3x + 6y = 0 \\ -3x + 6y = 0 \end{cases}$$

$$x = 2y$$

Let $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -x + 6y \\ -3x + 8y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix}$$

$$\begin{bmatrix} -6x + 6y \\ -3x + 3y \end{bmatrix} = 0$$

$$x = y$$

Let $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$X(t) = C_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Initial value condition : $x(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$X(0) = \begin{bmatrix} 2C_1 + C_2 \\ C_1 + C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$C_1 = 3$$

$$C_2 = -5$$

$$x(t) = 3e^{2t}\begin{bmatrix}2\\1\end{bmatrix} - 5e^{5t}\begin{bmatrix}1\\1\end{bmatrix}$$

Example 2

$$x' = Ax, \qquad A = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$$

$$p(\lambda) = \det\begin{pmatrix}2-\lambda & \\ & 2-\lambda\end{pmatrix}$$

$$= (\lambda - 2)^2$$

$$\lambda = 2$$

$$x(t) = C_1 e^{2t}\begin{bmatrix}1\\0\end{bmatrix} + C_2 e^{2t}\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \begin{bmatrix}C_1 e^{2t}\\ C_2 e^{2t}\end{bmatrix}$$

Example 3

$$x' = Ax, \qquad A = \begin{bmatrix}1 & -4\\4 & -7\end{bmatrix}$$

$$p(\lambda) = \det\begin{bmatrix}1-\lambda & -4\\ 4 & -7-\lambda\end{bmatrix}$$

$$= \lambda^2 + 6\lambda + 9$$

$$= (\lambda + 3)^2$$

$$\lambda = -3$$

$$A\begin{bmatrix}x\\y\end{bmatrix} = -3\begin{bmatrix}x\\y\end{bmatrix}$$

$$\begin{pmatrix}x - 4y\\ \end{pmatrix} \quad \begin{pmatrix}-3x\\ \end{pmatrix}$$

$$\begin{pmatrix} \\ 4x-7y \end{pmatrix} = \begin{pmatrix} \\ -3y \end{pmatrix}$$

$$\begin{pmatrix} 4x-4y \\ 4x-4y \end{pmatrix} = 0$$

$$x = y$$

Let $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$(A - \lambda I)\, w = v$$

$$\left( \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$4a - 4b = 1$$

Let $w = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$

$$x(t) = c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \left( e^{-3t} \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + t e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

$$= c_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \left( \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$