Method of Undetermined Coeff.

$$y'' + p(t)y' + q(t)y = g(t)$$

When $g(t)$ is "nice", we can make educated guesses about what the sol^n to ODE will look like.

Here "nice" usually means derivatives follow some nice pattern.

Before we begin: what does general sol^n to inhomogeneous eq^n look like?

<u>Thm:</u> The general sol^n to

$$y'' + p(t)y' + q(t)y = g(t)$$

looks like

$$y(t) = y_p(t) + y_h(t)$$

where $y_h(t)$ is the sol^n to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

<u>Proof:</u> If $y(t)$ is any solution, note $y - y_p$ is a sol^n to

$$y'' + p(t)y' + q(t)y = 0$$

$$\Rightarrow \quad y - y_p = \underbrace{c_1 y_1 + c_2 y_2}_{y_h(t)} \quad \text{for some } c_1, c_2$$

$$\Rightarrow \quad y = y_p(t) + y_h(t).$$

<u>Ex:</u> $\quad y'' - y' - 2y = 2e^{-2t}$

let's "guess" a sol^n: derivatives of e^{-2t} are exponential functions so we will try a solution of the form

$y_p(t) = Ae^{-2t}$ and solve for A.

$$y_p'' = 4Ae^{-2t} \qquad y_p' = -2Ae^{-2t}$$

$$4Ae^{-2t} + 2Ae^{-2t} - 2Ae^{-2t} = 2e^{-2t}$$

$$4Ae^{-2t} = 2e^{-2t}$$

$$\Rightarrow \quad A = 1/2$$

$$\Rightarrow \quad y_p = \frac{1}{2}e^{-2t}$$

If we want general sol^n:

y_h:
$$y'' - y' - 2y = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1)$$

$$y_h = C_1 e^{2t} + C_2 e^{-t}$$

$$y(t) = y_p(t) + y_h(t)$$

$$= \frac{1}{2}e^{-2t} + C_1 e^{2t} + C_2 e^{-t}$$

Ex: $y'' + 2y' - 3y = 5\sin(3t)$

Let's guess a solution of the form

$$y_p = A\sin(3t) + B\cos(3t)$$

$$y_p' = 3A\cos(3t) - 3B\sin(3t)$$

$$y_p'' = -9A\sin(3t) - 9B\cos(3t)$$

$$\left[-9A\sin(3t) - 9B\cos(3t)\right]$$

$$+\left[6A\cos(3t) - 6B\sin(3t)\right]$$

$$- 3A\sin(3t) - 3B\cos(3t)$$

$$= 5\sin(3t)$$

$$(-12A - 6B)\sin(3t)$$

$$+ (-12B + 6A)\cos(3t) = 5\sin(3t)$$

$$\begin{cases} -12A - 6B = 5 \\ 6A - 12B = 0 \end{cases}$$

$$\Longrightarrow \quad -30B = 5 \quad \Rightarrow B = -1/6$$

$$6A + 2 = 0 \quad \Rightarrow A = -1/3$$

$$y_p(t) = -\frac{1}{3}\sin(3t) - \frac{1}{6}\cos(3t).$$

Ex: $y'' + 2y' - 3y = 3t + 4$

Let's guess $y_p(t) = At + B$

$y_p' = A \qquad y_p'' = 0$

$$2A - 3(At + B) = 3t + 4$$

$$-3At + (2A - 3B) = 3t + 4$$

$$\begin{cases} -3A = 3 \\ 2A - 3B = 4 \end{cases}$$

$A = -1 \quad \Rightarrow \quad -2 - 3B = 4$

$\Rightarrow \quad B = -2$

$y_p(t) = -t - 2.$

<u>Ex:</u> $y'' - 2y' - 2y = e^{-2t} - 3e^{-t}$

To handle this, we'll use the following idea:

if y_1 solves

$y'' - 2y' - 2y = e^{-2t}$ and

y_2 solves

$y'' - 2y' - 2y = -3e^{-t}$

then $y_1 + y_2$ will solve

$$y'' - 2y' - 2y = e^{-2t} - 3e^{-t}$$

y_1 : $$y'' - 2y' - 2y = e^{-2t}$$

$$y_p(t) = Ae^{-2t}$$

$$y_p' = -2Ae^{-2t}$$

$$y_p'' = 4Ae^{-2t}$$

$$4Ae^{-2t} + 4Ae^{-2t} - 2Ae^{-2t} = e^{-2t}$$

$$\Rightarrow \quad 6Ae^{-2t} = e^{-2t}$$

$$\Rightarrow \quad A = 1/6$$

$$y_p = 1/6\, e^{-2t}$$

y_2: $y'' - 2y' - 2y = -3e^{-t}$

$y_p = Ae^{-t}$

$y_p' = -Ae^{-t}$

$y_p'' = Ae^{-t}$

$Ae^{-t} + 2Ae^{-t} - 2Ae^{-t} = -3e^{-t}$

$\Rightarrow A = -3$

$y_p = -3e^{-t}$

$y(t) = \frac{1}{6}e^{-2t} - 3e^{-t}$

Ex: $y'' - y' - 2y = e^{-t}$

$$y_p(t) = Ae^{-t}$$

$$y_p' = -Ae^{-t}$$

$$y_p'' = Ae^{-t}$$

$$\underbrace{Ae^{-t} + Ae^{-t} - 2Ae^{-t}}_{0} = e^{-t}$$

Our guess can't work out: there is no sol^n of the form Ae^{-t}!

New guess: $y_p(t) = Ate^{-t}$

$$y_p' = Ae^{-t} - Ate^{-t}$$

$$y_p'' = -Ae^{-t} - Ae^{-t} + Ate^{-t}$$

$$= -2Ae^{-t} + Ate^{-t}$$

$$y'' - y' - 2y = e^{-t}$$

$$(-2Ae^{-t} + Ate^{-t}) - (Ae^{-t} - Ate^{-t})$$

$$- 2Ate^{-t} = e^{-t}$$

$$-3Ae^{-t} = e^{-t}$$

$$A = -1/3$$

$$y_p(t) = -\frac{1}{3}te^{-t}$$

Common guesses:

- $e^{at} \rightsquigarrow Ae^{at}$

  if that doesn't work tack on powers of t:

  $Ate^{at}, \; At^2e^{at}, \; \ldots$

- $P(t)$ poly of degree d $\rightsquigarrow$ Polynomial of same degree:

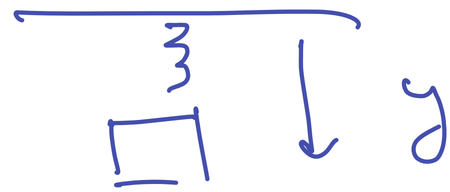
  $c_d t^d + \ldots + c_1 t + c_0$

- $\cos(at)$ or $\sin(at)$ $\rightsquigarrow$ $A\cos(at) + B\sin(at)$

$\vdots$

A brief application: Physics

Hooke's law:



$$F = -Ky$$

$$my'' = -Ky \leftarrow \text{Undamped}$$

Generally, need to account for friction.

$$my'' = -Ky - by'$$

$$y'' + py' + qy = 0$$

If we add external force to block, would get something like

$$y'' + py' + qy = f(t)$$

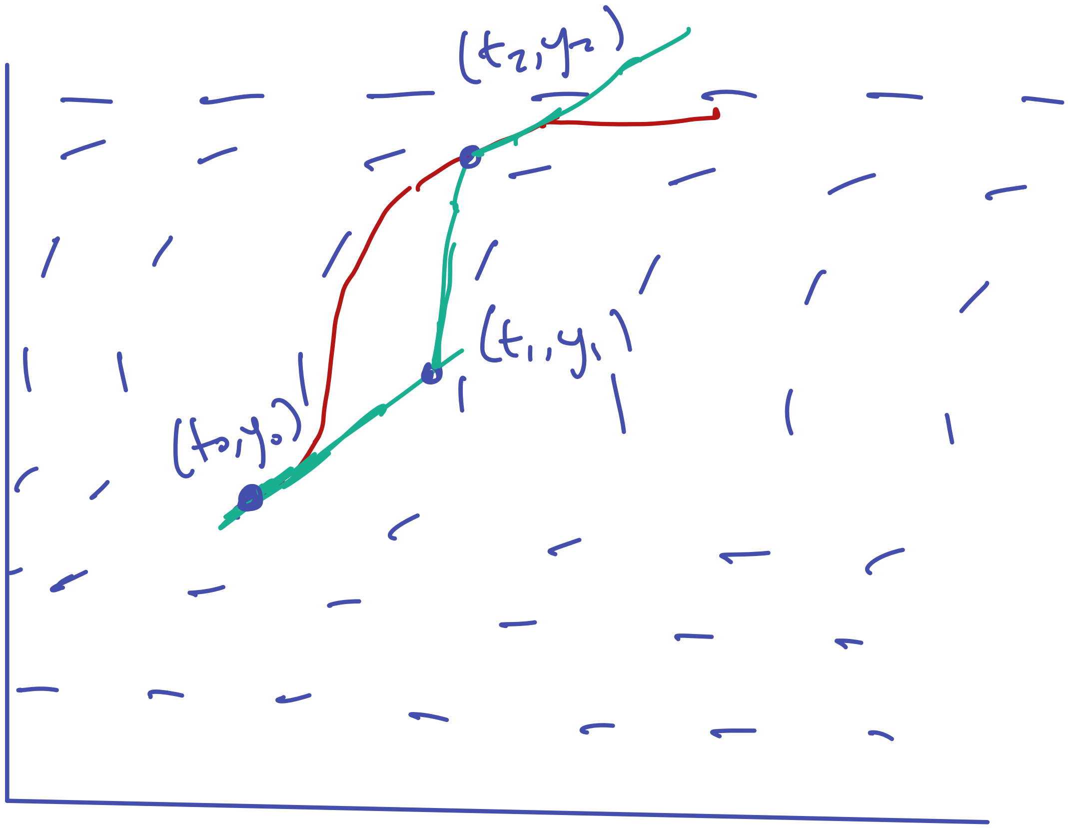
See 4.4 + 4.7 if you want a more indepth treatment of physical applications.

Euler's Method

How can we approximate the solution to an ODE if we can't solve it?

$$y' = f(t,y) \qquad y(t_0) = y_0$$

Recall that at each point (t,y) that $f(t,y)$ is the slope of tangent line to y at t.



Idea: tangent line approximation!

Choose some point (t_0, y_0).

Fix a step size Δt.

Take $t_1 = t_0 + \Delta t$

y_1 = pt on line through (t_0, y_0) of slope $f(t_0, y_0)$.

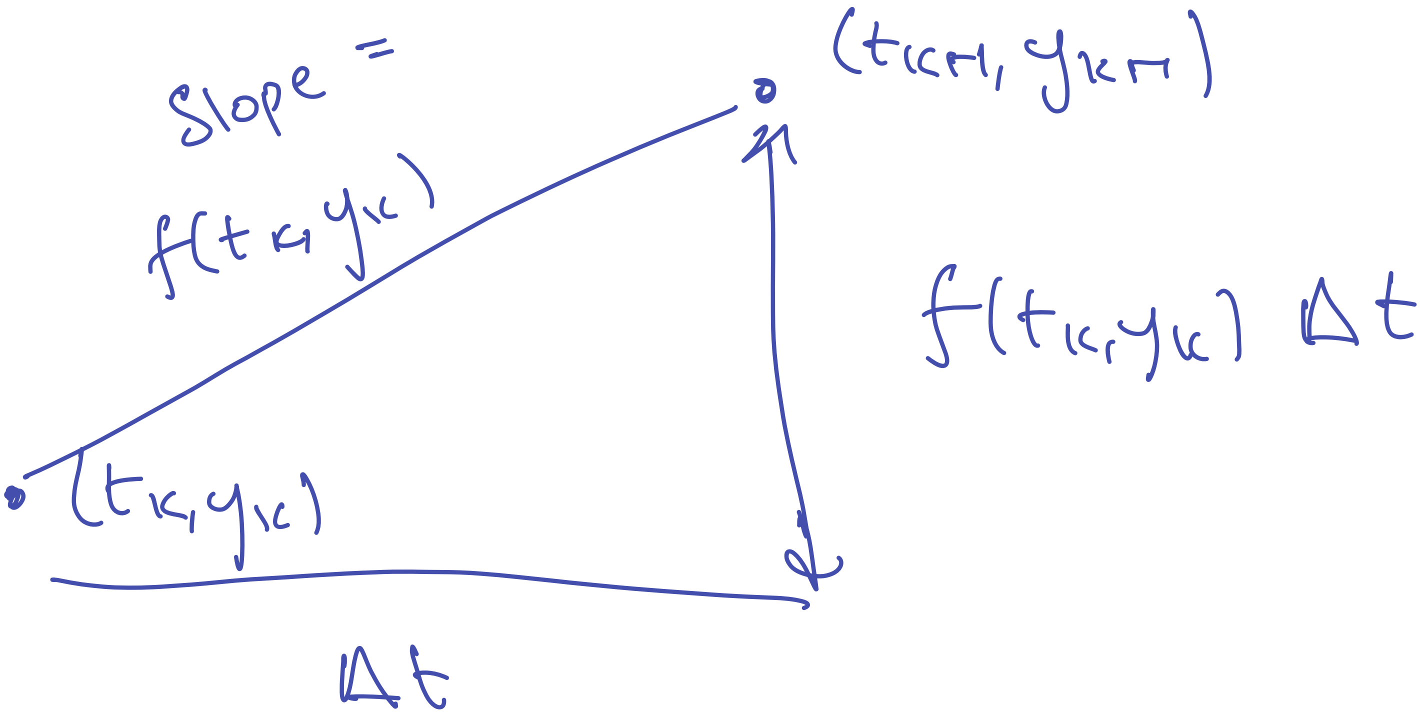
Keep repeating this procedure.

As Δt gets small, the points (t_k, y_k) are going to be "good" approximation to the path of the solution curve.

This procedure is called <u>Euler's Method</u>.

How can we come up with a recursive formula for this procedure?

$$t_{k+1} = t_k + \Delta t \quad \text{Easy!}$$



$$\frac{y_{k+1} - y_k}{t_{k+1} - t_k} = f(t_k, y_k)$$

$$\boxed{\begin{aligned} y_{k+1} &= f(t_k, y_k)\,\Delta t + y_k \\ t_{k+1} &= t_k + \Delta t \end{aligned}}$$

Ex: $y' = 2y - 1 \qquad y(0) = 1$

Let's approximate $y(1)$.

Let's start with $\Delta t = .1$

(this is pretty big).

$$t_{k+1} = t_k + .1$$

$$y_{k+1} = (2y_k - 1) \cdot .1 + y_k$$

$$= 1.2 y_k - .1$$

k	t_k	y_k
0	0	1
1	.1	1.1

2	.2	1.22
3	.3	1.364
4	.4	1.537
5	.5	1.744
⋮	⋮	⋮
10	1	3.596

$y(1) \approx 3.596$. How good of an approx. is this?

If you solve the IVP the solution is

$$y(t) = \frac{1}{2}(1+e^{2t}).$$

$$y(1) = \frac{1}{2}(1+e^{2}) \approx 4.195$$

So our approximation was not great.

Let's make Δt smaller.

If we take $\Delta t = .01$ after 100 iterations, we'll get

$$y_{100} \approx 4.1223.$$

Ex: $y' = -2ty^2 \qquad y(0) = 1.$

Suppose we wanted to approximate $y(2)$.

Start with $\Delta t = .5$

(pretty large!)

$$t_{k+1} = t_k + .5$$

$$y_{k+1} = \left(-2t_k y_k^2\right) \cdot \frac{1}{2} + y_k$$

$$= -t_k y_k^2 + y_k$$

$$= y_k \left(1 - t_k y_k\right)$$

k	t_k	y_k
0	0	1
1	.5	1

2	1	1/2
3	1.5	1/4
4	2	5/32

$y(2) \approx 5/32$. Let's see how we did:

$y(t) = 1/1+t^2$ is explicit sol^n.

$y(2) = 1/5$

Again, not amazing.

$\Delta t = .001$, after 2000 steps we'd get

$y_{2000} \approx .199937$

Not terrible.

We'll do some error analysis at a later date.

How to deal with higher order ODEs?

$$y'' = F(t, y, y')$$

Let's set $v = y'$

$$v' = y'' = F(t, y, v)$$

This gives us a system of first order ODEs:

$$\begin{cases} y' = v \\ v' = F(t, y, v) \end{cases}$$

We can use Euler's method to approx. y and v.

$$t_{k+1} = t_k + \Delta t$$

$$y_{k+1} = v_k \Delta t + y_k$$

$$v_{k+1} = F(t_k, y_k, v_k)\Delta t + v_k$$

Ex: $y'' + y = 0 \qquad y(0) = -1$

$\qquad\qquad\qquad\quad y'(0) = -2$

$v = y'$

$v' = y'' = -y$

$$\begin{cases} y' = v & y(0) = -1 \\ v' = -y & v(0) = -2 \end{cases}$$

$\Delta t = .1$

$t_{k+1} = t_k + .1$

$y_{k+1} = v_k \cdot .1 + y_k$

$v_{k+1} = -y_k \cdot .1 + v_k$

k	t_k	y_k	v_k
0	0	-1	-2
1	.1	-1.2	-1.9
2	.2	-1.39	-1.78
$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	1	-2.34	$-.259$

$y(1) \approx -2.34$

Can solve ODE explicitly:

$$y'' + y = 0$$

$$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

$$y(t) = -\cos(t) - 2\sin(t)$$

$$y(1) = -\cos(1) - 2\sin(1)$$

$$\approx -2.223$$

If we use $\Delta t = .01$

$$y(1) \approx -2.224$$

For systems with more equations, can just do Euler's method in each variable.

This idea is the key to studying ODEs:

Higher order linear ODEs $\rightsquigarrow$ System of first order ODEs

The "right" way to study ODEs is by studying systems of ODEs.