Math 33 B : 7/7

Second order linear differential equations
(with constant coefficients)

$$y' + p(t)\, y = q(t)$$

$$y'' + p(t)\, y' + q(t)\, y = r(t)$$

$$y'' + p y' + q y = 0$$

Strategy $\quad e^{\lambda t}$

$$\lambda^2 + p\lambda + q = 0$$

2 distinct real roots $\qquad e^{\lambda_1 t} \qquad e^{\lambda_2 t}$

1 repeated real root $\qquad e^{\lambda t} \qquad t e^{\lambda t}$

2 distinct complex roots

$\lambda_1 = a + bi \qquad e^{\lambda_1 t} \qquad e^{\lambda_2 t}$

$\lambda_2 = a - bi \qquad e^{at}\cos bt \qquad e^{at}\sin bt$

Example 1 $\quad 10y'' - y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 0$

$$10\lambda^2 - \lambda - 3 = 0$$

$$\lambda^2 - \frac{1}{10}\lambda - \frac{3}{10} = 0$$

$$\left(\lambda - \frac{3}{5}\right)\left(\lambda + \frac{1}{2}\right) = 0$$

$$\lambda = \frac{3}{5} \quad \text{or} \quad \lambda = -\frac{1}{2}$$

2 distinct real solutions

$$y(t) = Ae^{\frac{3}{5}t} + Be^{-\frac{1}{2}t}$$

$y(0) = 1$, $y'(0) = 0$ $\Rightarrow$ $A + B = 1$

$$y'(t) = \frac{3}{5}Ae^{\frac{3}{5}t} - \frac{B}{2}e^{-\frac{1}{2}t}$$

$$\Rightarrow \quad \frac{3}{5}A - \frac{B}{2} = 0$$

$$\frac{1}{2}A + \frac{1}{2}B = \frac{1}{2}$$

$$\frac{11}{10}A = \frac{1}{2}$$

$$A = \frac{5}{11}$$

$$B = \frac{6}{11}$$

$$y(t) = \frac{5}{11}e^{\frac{3}{5}t} + \frac{6}{11}e^{-\frac{1}{2}t}$$

Example 2

$$16y'' - 8y' + y = 0, \quad y(0) = 4, \quad y'(0) = -2$$

$$16\lambda^2 - 8\lambda + 1 = 0$$

$$(4\lambda - 1)^2 = 0$$

$$\lambda = \frac{1}{4}$$

1 repeated real solution

$$y(t) = Ae^{\frac{1}{4}t} + Bte^{\frac{1}{4}t}$$

$$y(0) = 4$$

$$A = 4$$

$$y'(0) = -2$$

$$y'(t) = \frac{A}{4}e^{\frac{1}{4}t} + B\left(e^{\frac{1}{4}t} + \frac{1}{4}te^{\frac{1}{4}t}\right)$$

$$\frac{A}{4} + B = -2$$

$$B = -3$$

$$y(t) = 4e^{\frac{1}{4}t} - 3te^{\frac{1}{4}t}$$

$$= e^{\frac{1}{4}t}(4 - 3t)$$

complex exponential functions

$$e^t \qquad e^{x+iy} = e^x e^{iy}$$

$$= e^x(\cos y + i \sin y)$$

$$e^{2-\frac{\pi}{2}i} = e^2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$= -e^2 i$$

Example 3 $\quad 9y'' + y = 0, \quad y(3\pi) = 2, \quad y'(3\pi) = -2$

$$9\lambda^2 + 1 = 0$$

$$\lambda^2 = -\frac{1}{9}$$

2 distinct complex

$$\lambda = \pm \frac{1}{3} i$$

solutions

$$y(t) = A \cos \frac{t}{3} + B \sin \frac{t}{3}$$

$$y(3\pi) = 2$$

$$A \cos\left(\frac{3\pi}{3}\right) + B \sin\left(\frac{3\pi}{3}\right) = 2$$

$$-A = 2$$

$$A = -2$$

$$y'(3\pi) = -2$$

$$y'(t) = -\frac{A}{3} \sin \frac{t}{3} + \frac{B}{3} \cos \frac{t}{3}$$

$$y'(3\pi) = -\frac{A}{3} \sin \pi + \frac{B}{3} \cos \pi = -2$$

$$-\frac{B}{3} = -2$$

$$B = 6$$

$$y(t) = -2 \cos\left(\frac{t}{3}\right) + 6 \sin\left(\frac{t}{3}\right)$$

Example 4 $\quad y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 3$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= -1 \pm i$$

$$y(t) = Ae^{-t}\cos t + Be^{-t}\sin t$$

$$y(0) = 2 \Rightarrow A = 2$$

$$y'(0) = 3$$

$$y'(t) = A(-e^{-t}\cos t - e^{-t}\sin t)$$
$$+ B(-e^{-t}\sin t + e^{-t}\cos t)$$

$$-A + B = 3$$
$$B = 5$$

$$y(t) = 2e^{-t}\cos t + 5e^{-t}\sin t$$
$$= e^{-t}(2\cos t + 5\sin t)$$

$$\frac{dP}{dt} = 0.75\left(1 - \frac{P}{350}\right)P$$

$$\frac{dP}{dt} = CP$$

$$\frac{dP}{dt} = C_1(\underline{1 - C_2 P})P$$

L

$$\frac{dP}{dt} = 0.75\left(1 - \frac{P}{350}\right)P - \frac{3}{365}L$$