Application: Population growth

$P(t)$ population at time t

$\beta(t)$ Birth rate $\leftarrow$ $\dfrac{\text{unit of population}}{\text{unit of time}}$

$\delta(t)$ Death rate

In some interval $[t, t+\Delta t]$

$\#$ Births $= \beta(t) \cdot P(t) \cdot \Delta t$

$\#$ Death $= \delta(t) \cdot P(t) \cdot \Delta t$

$$\Delta P = (\beta(t) - \delta(t))\Delta t$$

$$\frac{\Delta P}{\Delta t} = (\beta(t) - \delta(t))$$ So as $\Delta t \to 0$

$$\frac{dP}{dt} = (\beta - \delta)P$$ General Population equation

This is separable and so in theory we can solve.

In the real world, growth usually bounded, lack of food, space, etc.

Let's assume β dec. linearly and δ inc. linearly:

$$\beta = \beta_0 - \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

$$\frac{dP}{dt} = (\beta_0 - \delta_0) P - (\beta_1 + \delta_1) P^2$$

$$r = \beta_0 - \delta_0$$
$$K = r / \beta_1 + \delta_1$$

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P$$

Logistic growth model

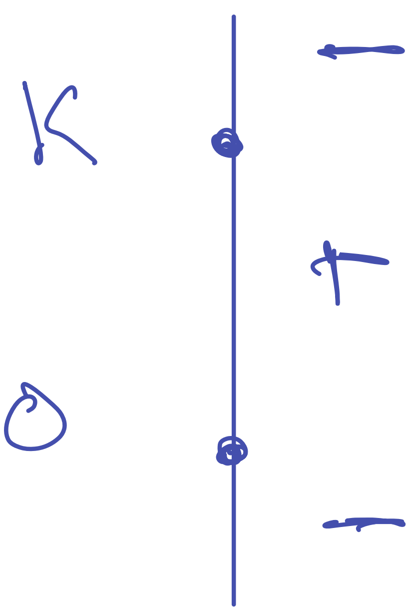
r = growth rate

K = Carrying capacity

This is autonomous, so let's do qualitative analysis:

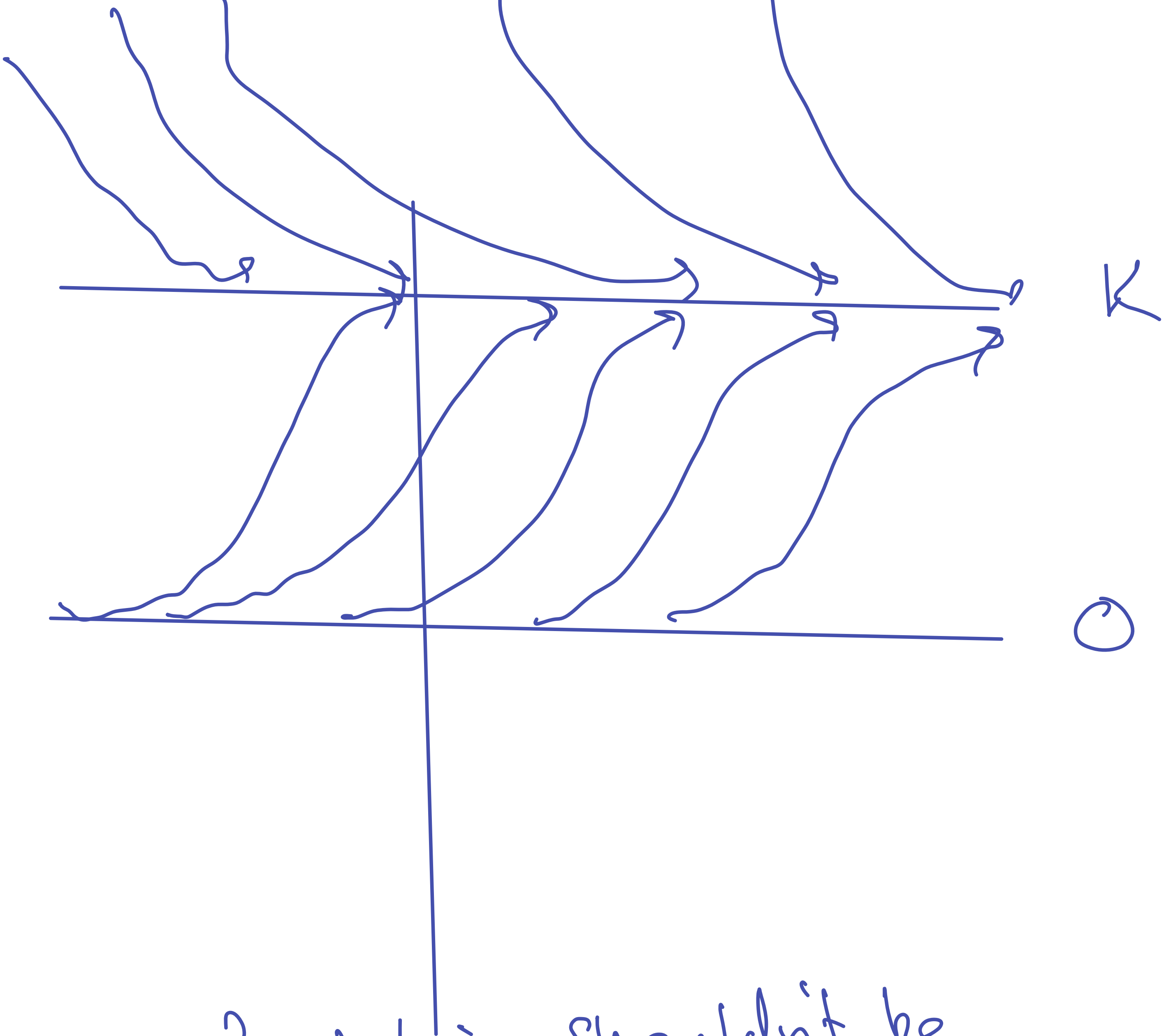
Equilibrium points: $r\left(1-\frac{P}{K}\right)P = 0$

$P=0$ and $P=K$



$P=K$ stable

$P=0$ unstable



Note: Population shouldn't be negative so don't care about those graphs.

We see $\lim_{t\to\infty} P(t) = K$

which is why it's called the carrying capacity.

Explicit sol^n:

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P$$

$$K\frac{dP}{dt} = r(K-P)P$$

$$\frac{K}{(K-P)P}\,dP = r\,dt$$

Note that $\dfrac{K}{(K-P)P} = \dfrac{1}{P} + \dfrac{1}{K-P}$

$\Longrightarrow$

$$\int \frac{1}{P} + \frac{1}{K-P}\,dP = \int r\,dt$$

$$\ln|P| - \ln|K-P| = rt + C$$

$$\Longrightarrow \quad \frac{P}{K-P} = Ce^{rt}$$

$$P(t) = \frac{KCe^{rt}}{1+Ce^{rt}}$$

If we let $P_0 = P(0)$

$$P_0 = \frac{KC}{1+C} \Longrightarrow C = \frac{P_0}{K-P_0}$$

$$\boxed{P(t) = \frac{KP_0}{P_0 + (K-P_0)e^{-rt}}}$$

One can easily then see that

$$\lim_{t \to \infty} P(t) = K.$$

Ex: Suppose we have a pond that can support 1000 fish. Let's say you have 100 fish initially in the pond. Suppose after 1 year, there are 200 fish in the pond. How long to hit 90% capacity?

$K = 1000$ $\quad P_0 = 100$

$P(1) = 200$

$$P(t) = \frac{1000 \cdot 100}{900 e^{-rt} + 100}$$

$\Rightarrow$

$$200 = \frac{1000 \cdot 100}{900e^{-r} + 100} \Rightarrow$$

$$r = \ln(9/4)$$

$$900 = P(t)$$

$$900 = \frac{1000 \cdot 100}{900e^{-rt} + 100}$$

$$\Rightarrow \quad t \approx 5.419 \text{ years}$$

Harvesting: Suppose we have a pond and we're trying to model population of fish in the pond.

Suppose our model is

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right)$$

Let's say we allow for H fish to be fished each year.

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - H$$

Let's start by investigating behavior for certain values of H.

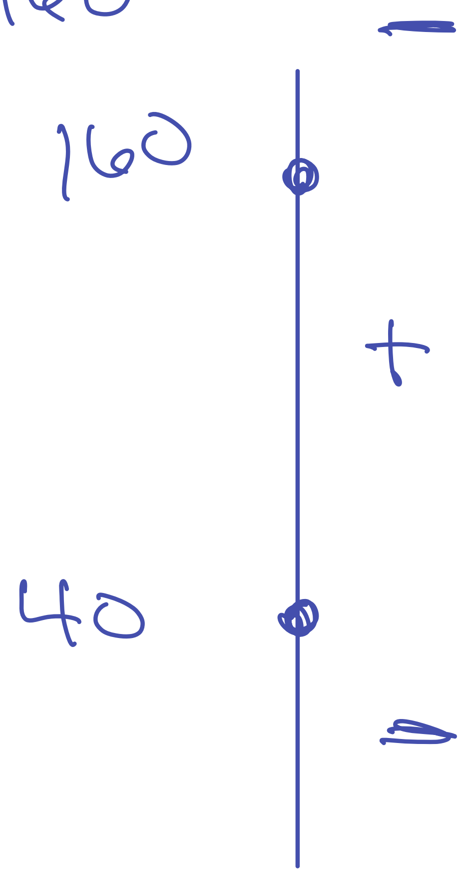
$H=32$: $$\frac{dP}{dt} = P\left(1-\frac{P}{200}\right) - 32$$

Equilibrium pts: $P\left(1-\frac{P}{200}\right)-32=0$

$P = 40, 160$

40 unstable

160 stable



If P drops below 40, the population will die off!

$H = 100$:

$$\frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - 100$$

Equilibrium pts: $P\left(1 - \frac{P}{200}\right) - 100 = 0$

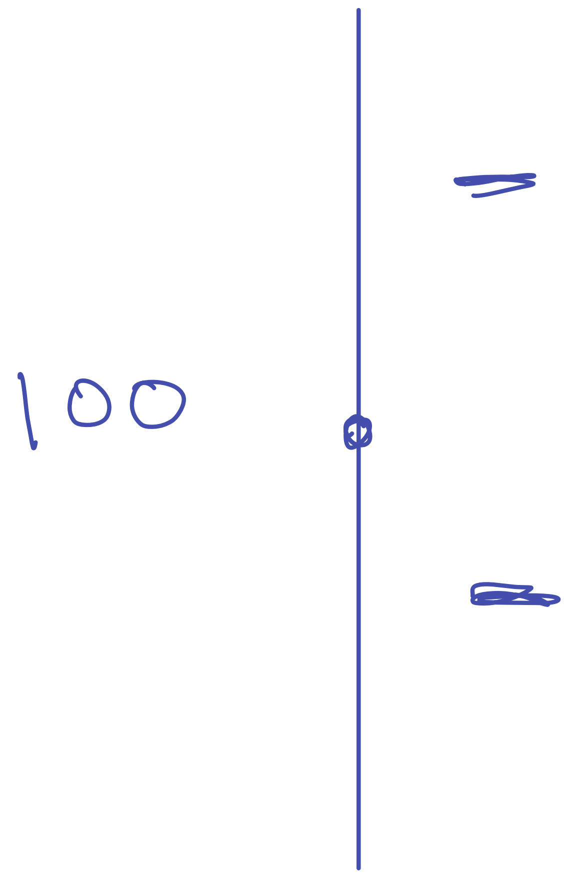
No real sol^{n}s!

Note that $P\left(1 - \frac{P}{200}\right)$ has maximum value of 50, and so $\frac{dP}{dt} < 0$

$\Rightarrow$ population always dies off!

$H = 50$: $\quad \frac{dP}{dt} = P\left(1 - \frac{P}{200}\right) - 50$

Equilibrium pts: $\quad P = 100$



Population good as long as $P \geq 100$.

What's going on?

$$\frac{dP}{dt} = rP(1-P/K) - H$$

Equilibrium pts:

$$rP(1-P/K) - H = 0$$

$$-rP^2 + rKP - KH = 0$$

$$\Rightarrow \quad P = \frac{K}{2} \pm \sqrt{\frac{K^2}{4} - \frac{KH}{r}}$$

$$\Delta = \frac{K^2}{4} - \frac{KH}{r}$$

$\Delta < 0$: $\quad H > rK/4$

no real sol^n $\Rightarrow$ no equil.

pts and $\frac{dP}{dt} < 0$

$\frac{d}{dt}$

So population is doomed!

$\Delta > 0$: $H < rK/4$ two

equilibrium pts:

$\frac{K}{2} + \sqrt{\Delta}$ stable

$\frac{K}{2} - \sqrt{\Delta}$ Unstable

$\Delta = 0$: $H = \frac{rK}{4}$

$\frac{K}{2}$ Semistable

The behavior depends drastically on the parameter H.

In our case.. $H = 50$ is "special": we call it a <u>bifurcation point</u> because # equil pts changes from 2 to 1 to 0 as we pass through $\cancel{H}$.

In general:

$$y' = f_\lambda(y)$$

$f_\lambda(y)$ depends on some parameter λ

One parameter family of ODEs

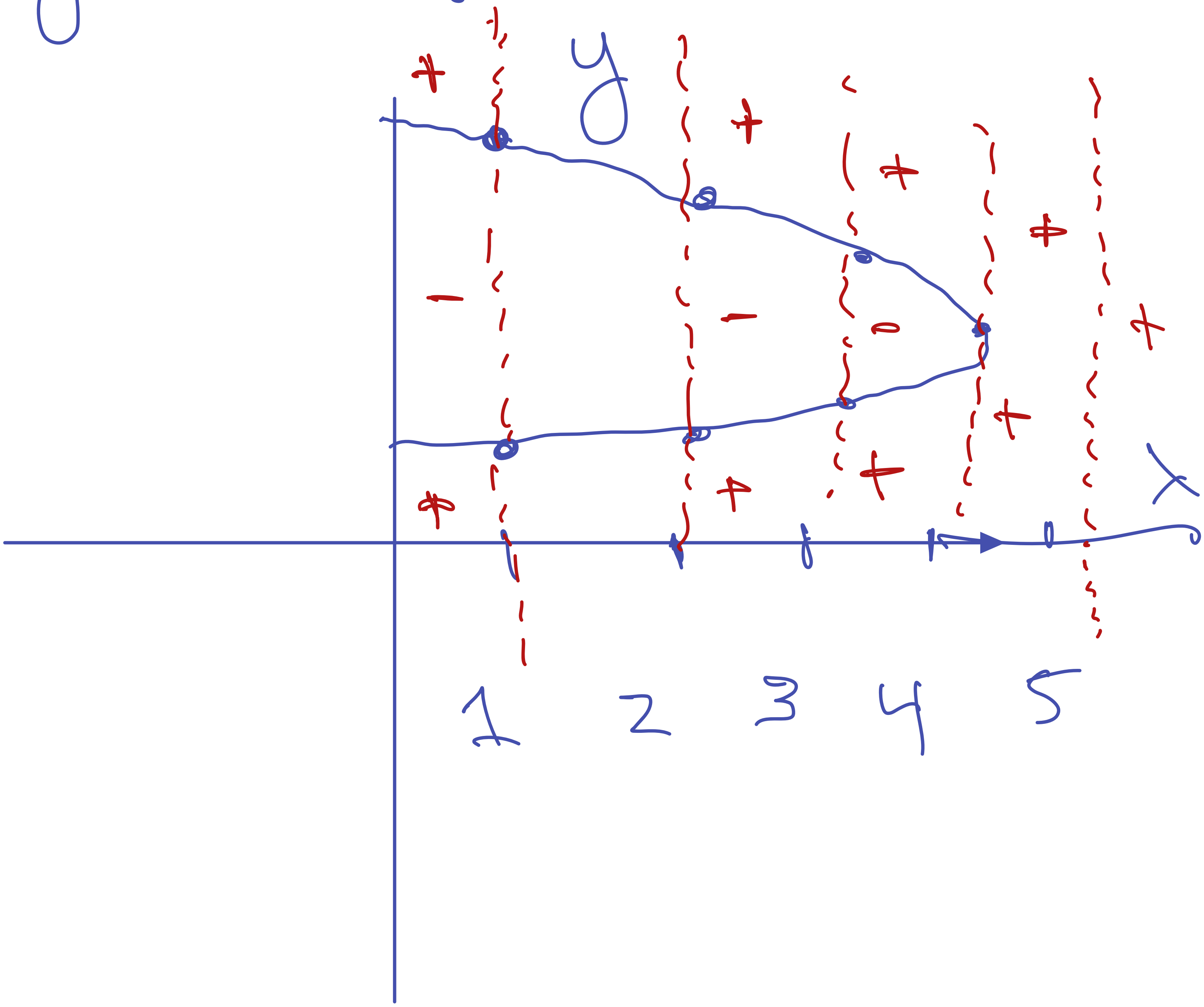
bifurcation at $\lambda = \lambda_0$ if # equilibrium pts changes

Usually analyze this with something called a <u>bifurcation diagram</u>.

<u>Ex:</u> $y' = f_\lambda(y) = y^2 - 4y + \lambda$

The equil. solutions are

$$y = 2 \pm \sqrt{4-\lambda}$$

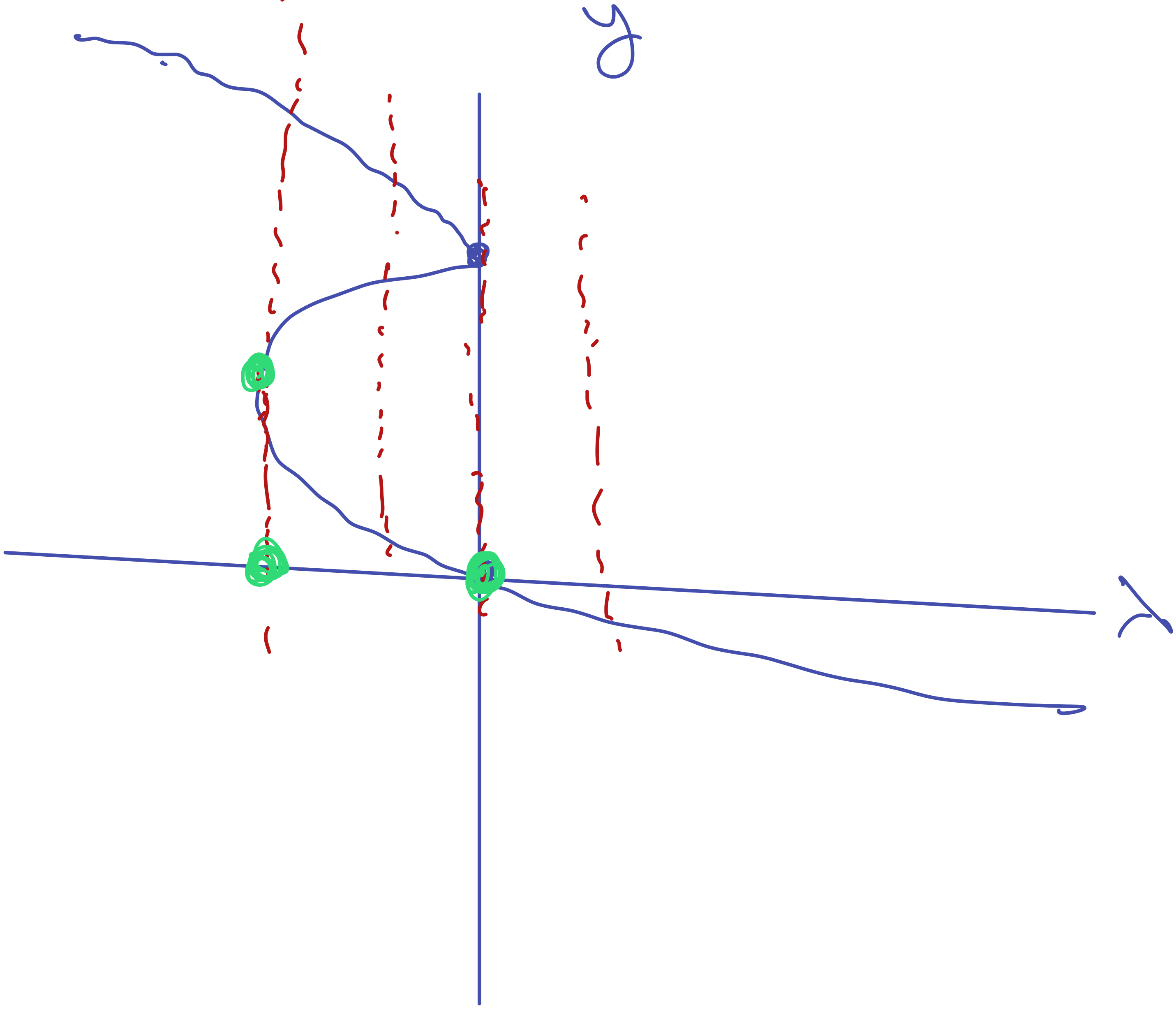


We plot $y^2 - 4y + \lambda = 0$ in the λ-y plane. For each choice of λ, we see how may equil. pts there are and if we want can figure out phase diagram at each pt.

We can see at $\lambda = 4$ we have bifurcation b/c change from 2 to 1 to 0 equilibrium pts.

Ex: $y' = f_\lambda(y) = y(y-2)^2 + \lambda$

Bifurcation diagram:



$$y(y-2)^2+\lambda = 0$$

We have two bifurcation values. How to find them?

The left most bifurcation will happen when

$$f_\lambda'(y) = 0.$$

$$f_\lambda'(y) = 3y^2 - 8y + 4 = 0$$

$y = 2$ $\qquad\qquad$ $y = 2/3$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

$\lambda = 0$ $\qquad\qquad$ $\frac{2}{3}\left(\frac{4}{3}\right)^2 + \lambda = 0$

$$\lambda = -32/27$$

In general, y value where bifurcation happens: $f_\lambda'(y) = 0$

Plug into equation to find bifurcation value λ.

Second order ODEs

$$y'' = f(t, y, y')$$

a sol^n is a function $y(t)$ that is twice continuously differentiable

Many problems in physics give rise to second order ODEs

e.g. $\vec{F} = m\vec{a}$, harmonic oscillation, etc.

We're mostly going to focus on linear 2^{nd} order ODEs:

$$y'' + p(t)\, y' + q(t)\, y = g(t)$$

if $g(t) = 0$, we call such an equation homogenous.

Def: A linear combination of two functions y_1 and y_2 is an expression of the form

$$c_1 y_1 + c_2 y_2 \quad \text{for } c_1, c_2 \in \mathbb{R}.$$

Prop: If y_1 and y_2 both solve

$$y'' + p y' + q y = 0$$

Then any linear combination of y_1 and y_2 will solve this diff eq.

Proof: Exercise.

Def: y_1 and y_2 are called <u>linearly dependent</u> if

$y_1 = K y_2$ for some some constant K. If y_1 and y_2 are not linearly dependent, they are called <u>linearly independent</u>

Ex: $y_1(t) = t \qquad y_2(t) = t^2$

$y_1(t)$ and $y_2(t)$ are linearly independent on $\mathbb{R}$.

$y_1(t) = \sin(t) \qquad y_2(t) = -4\sin(t)$

$y_1(t)$ and $y_2(t)$ are linearly dependent

Thm: y_1 and y_2 are linearly independent sol^n to

$y'' + py' + qy = 0$ then

the general sol^n is

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for arbitrary c_1, c_2.

Note: • we study linear ODEs b/c we can use techniques of linear algebra. The sol^n set to a 2^{nd} order homogenous differential eq^n forms a 2-dim vector space, so 2 linearly independent vectors yield a basis.

• We still have existence and uniqueness thm for 2nd order ODE:

$$y'' + py' + qy = g$$

$$y(0) = y_0$$
$$y'(0) = y_0'$$

will have unique solution to IVP on some interval given it's sufficiently "nice"

How to tell if two functions are linearly independent?

Def: The <u>Wronskian</u> $W(y_1, y_2)$

of y_1, y_2 is defined by

$$W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

$$= y_1 y_2' - y_2 y_1'$$

The Wronskian measures how linearly independent two functions are:

<u>Thm</u>: y_1, y_2 are linearly dependent on I. Then $W(y_1, y_2) \equiv 0$ on I.

Alternatively, $W \not\equiv 0$ on I, then y_1, y_2 are linearly independent.

Proof: $y_1 = k y_2$

$y_1' = k y_2'$

$W(y_1, y_2) = (k y_2) y_2' - y_2 (k y_2') = 0$ $\square$

Warning: $W \equiv 0 \not\Rightarrow y_1, y_2$ linearly dependent in general!!

Thm: if y_1, y_2 are sol^n to

$$y'' + p y' + q y = 0 \text{ on } I,$$

then $W(y_1, y_2) \equiv 0$ on I

$\Longleftrightarrow$ y_1, y_2 linearly dependent.

Proof:

$\Leftarrow$) Always holds from above thm.

$\Rightarrow$) Suppose $W(y_1, y_2) \equiv 0$ on I. If $y_2 \equiv 0$ on I, then $y_2 = 0 \cdot y_1 \Rightarrow$ linearly dependent. Otherwise $y_2(t_0) \neq 0$ for some $t_0 \in I$.

By continuity, $y_2(t) \neq 0$

in some interval I_0 containing t_0. Then note

$$\left(\frac{y_1}{y_2}\right)' = \frac{W(y_1, y_2)}{y_2^2} = 0$$

$\Rightarrow$ $y_1 = K y_2$ for some $K \in \mathbb{R}$ on I_0. Since y_1 and $K y_2$ solve the same ODE on I_0 by uniqueness, they must agree on all of I

$\Rightarrow$ y_1 and y_2 linearly dependent on I_{π}.

Ex: $y'' + 4\sin(y) = 0$

$y(0) = 4$

$y'(0) = 2$

Can check that

$y_1(t) = \cos(2t)$ are both

$y_2(t) = \sin(2t)$ sol^n

$$W(y_1, y_2) = \det\begin{pmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{pmatrix}$$

$$= 2\cos(2t)^2 + 2\sin(2t)^2 = 2.$$

So $W \neq 0 \implies$

y_1, y_2 linearly independent

$$\Rightarrow \quad y(t) = C_1 \cos(2t) + C_2 \sin(2t)$$

1) general sol^n: