Existence and uniqueness

So far, we've seen ways to solve IVP for first order ODE. How can we know if a sol^n exists if we can't say, find an integrating factor?

Ex: $y' = \frac{1}{t} \quad y(0) = 0$

General solution is given by

$$y(t) = \ln|t| + C$$

not defined at $t=0$, so a sol^n

might not always exist.

Ex: $y' = 2\sqrt{y} \qquad y(0) = 0$

$$\frac{1}{\sqrt{y}} y' = 2 \implies 2\sqrt{y} = 2t + C$$

$$\implies C = 0 \implies y = t^2.$$

However, note that the constant function $y(t) = 0$ is also a solution to the IVP

Natural questions:

- When will a solution exist?

Existence

- How many solutions will there be?

Uniqueness

For first order ODEs, we can answer these questions.

$$y' = f(t,y) \quad \text{general first order ODE.}$$

Assume that $f(t,y)$ is defined in some rectangle R.

$$a < t < b \qquad R$$
$$c < y < d$$

<u>Thm:</u> (Existence and uniqueness theorem)

$f(t,y)$ defined R and is cts on R.

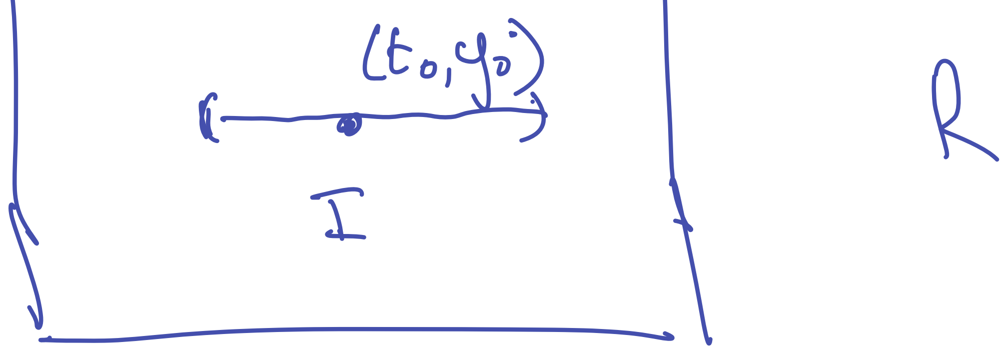
Then exists a solution $y(t)$ to

$$y' = f(t,y) \quad \text{with} \quad y(t_0) = y_0$$

for any $(t_0, y_0) \in R$, and

$y(t)$ is defined in some interval containing t_0.

Furthermore, if $\frac{df}{dy}$ is continuous on R, then there is a unique solution to the IVP $y' = f(t,y)$, $y(t_0) = y_0$ for $(t_0, y_0) \in R$ with $y(t)$ defined on some interval containing t_0.



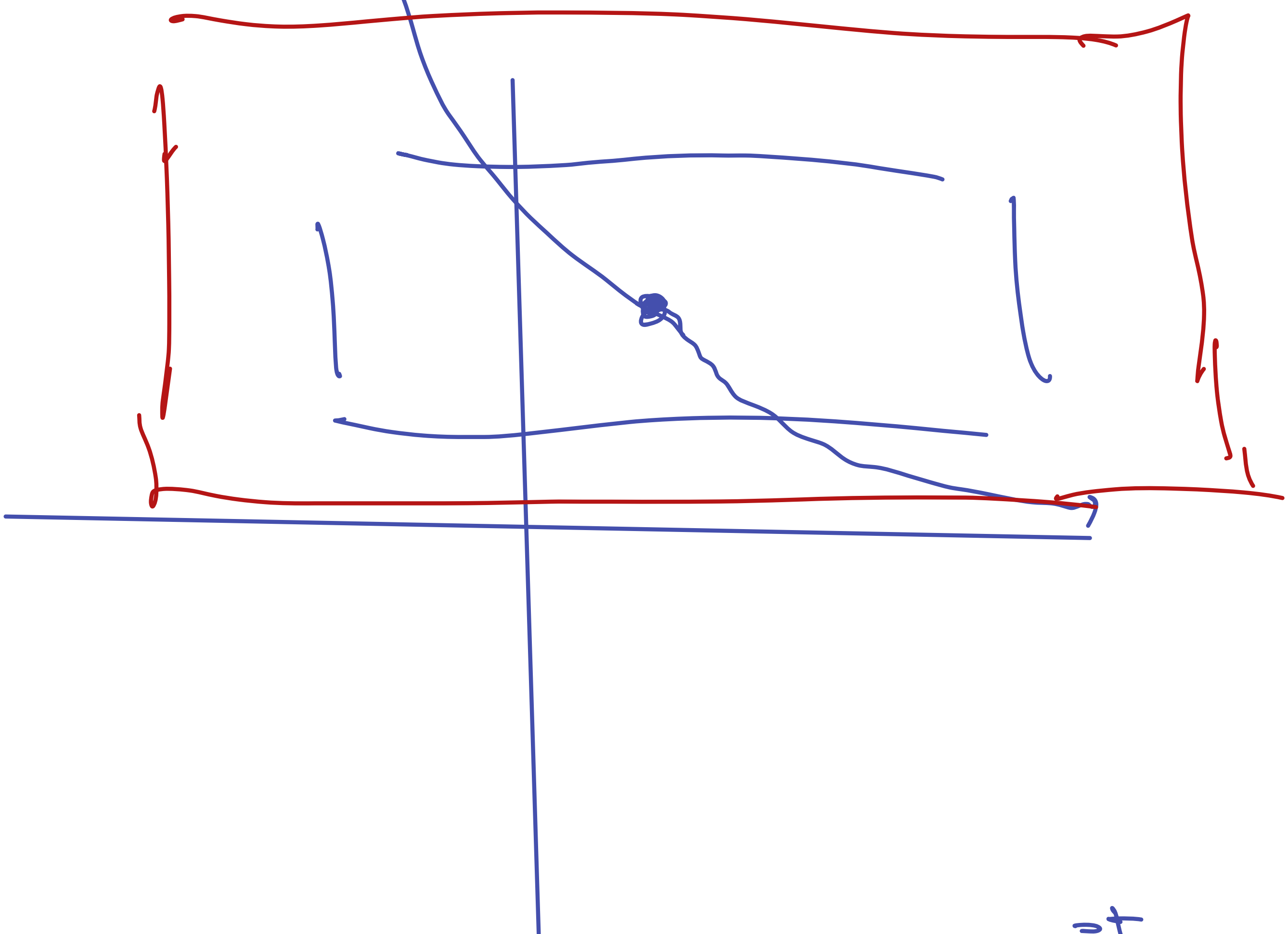
Ex: $y' = -y$.

$f(t,y) = -y$. This is cts everywhere

$\frac{\partial f}{\partial y} = -1$ This cts everywhere

Given any (t_0, y_0) and any rectangle R containing this point, there is a unique solution to $y' = -y$, $y(t_0) = y_0$.

We know that solutions look like $y(t) = Ce^{-t}$



Note that $y(t) = Ce^{-t}$ is defined everywhere, but the thm is only guaranteeing existence on some interval around t_0.

Ex: $y' = -2\sqrt{y}$ $\quad y(0) = 0$

$f(t,y) = -2\sqrt{y}$ cts for $y>0$

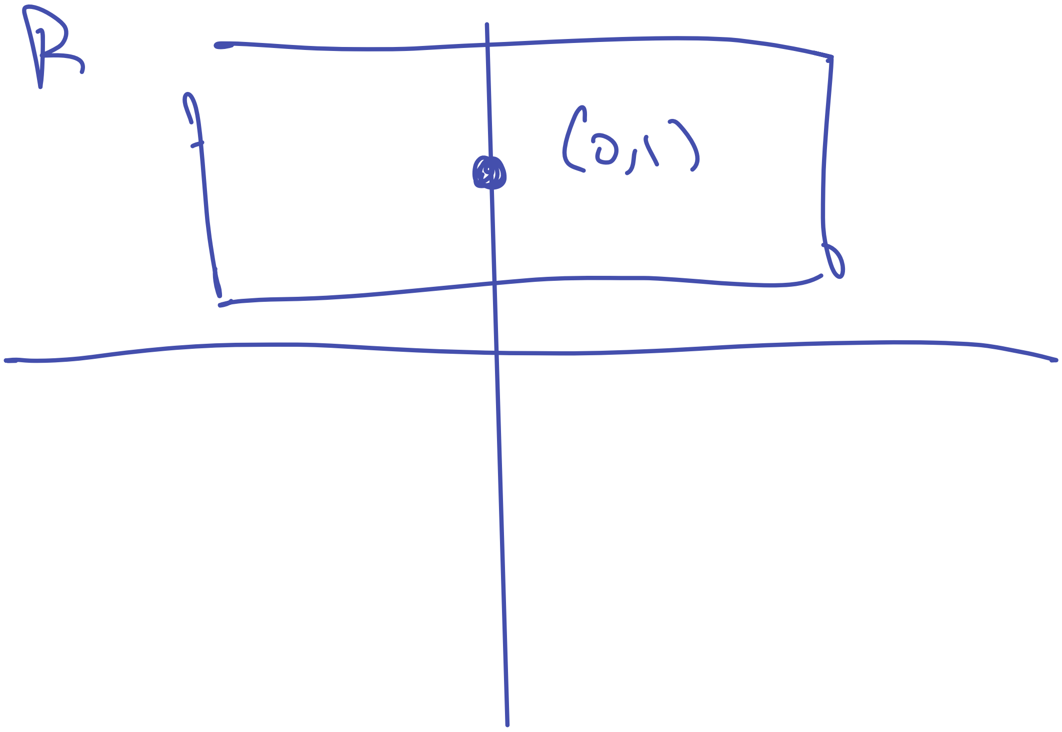
$\frac{df}{dy} = -\frac{1}{\sqrt{y}}$ not defined at $y=0$

Because for any R containing $(0,0)$ both f and $\frac{df}{dy}$ will not be cts, we can't apply the theorem, so neither uniqueness nor existence are guaranteed.

That's why it was possible to find two different solutions to the IVP.

$y(0) = 1$

R



However, the IVP with $y(0) = 1$ will have a unique soln

$$2\sqrt{y} = 2t + C \qquad y = (t + 1/4)^2$$

$$2\sqrt{y} = 2t + 1/2$$

Ex: $ty' = 2y \quad y(1) = -1$

$f(t,y) = \frac{2y}{t}$

$\frac{\partial f}{\partial y} = \frac{2}{t}$

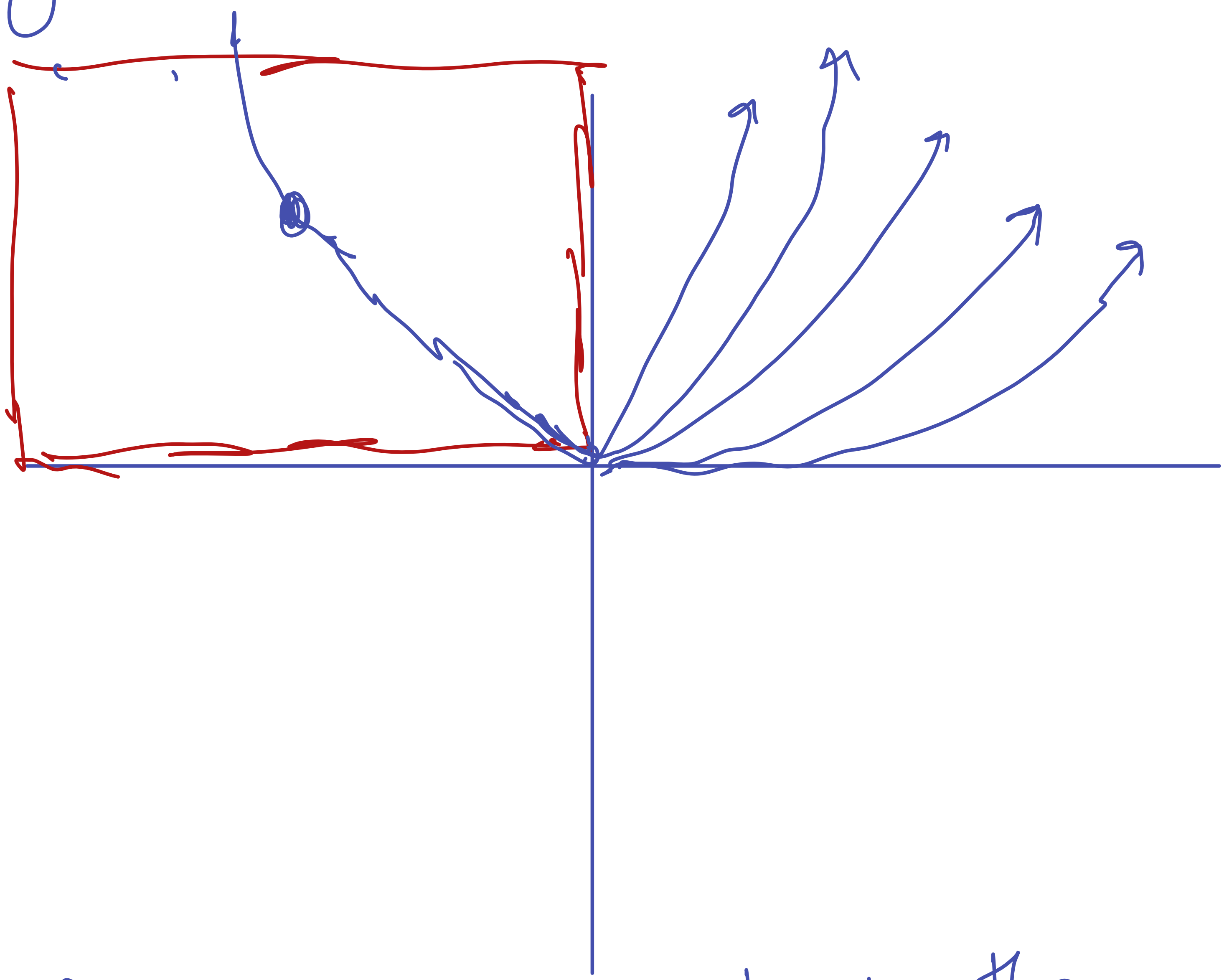
both well defined and cts away from $t = 0$.

$R = (-2,0) \times (0,2)$

there is a unique sol^n to this IVP on some interval inside R.

$$y(t) = \begin{cases} t^2 & t \le 0 \\ Ct^2 & t > 0 \end{cases}$$

for arbitrary C, then $y(t)$ is a sol^n to IVP.



Outside the rectangle, the

Curves could branch off, but once you enter they have to agree b/c of the theorem.

Ex: $y' = y^2$ $\quad y(0) = 1$

$f(t,y) = y^2$

$\frac{df}{dy} = 2y$

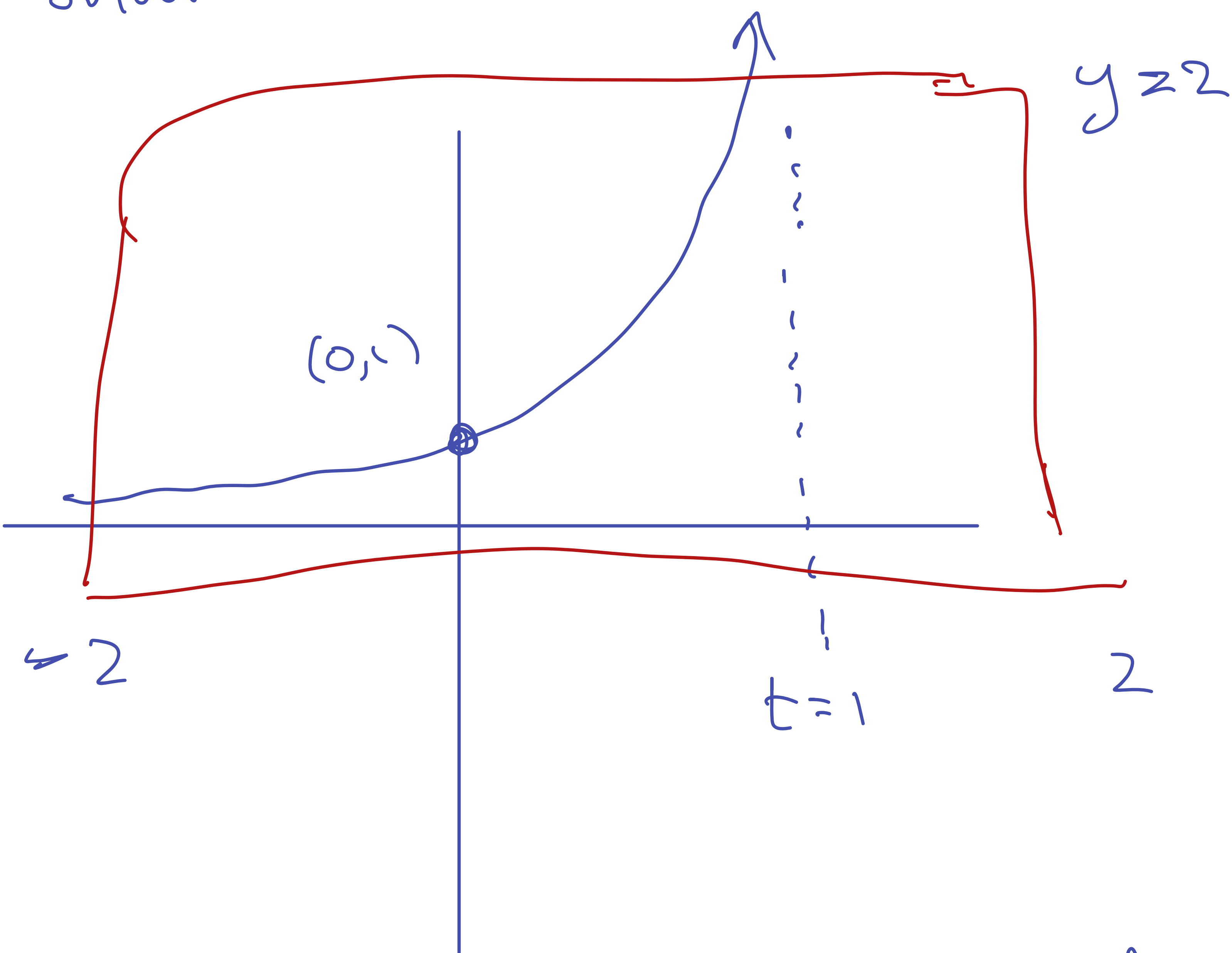
these are all defined everywhere

$R = (-2,2) \times (0,2)$

there is a unique sol^n to

IVP inside R.

$y(t) = 1/1-t$ is the solution to the IVP.



the sol^n is only guaranteed to exist/be unique somewhere

Inside $(-2,1)$.

But actual interval of existence is $(-\infty,1)$.

Geometrically, the existence and uniqueness thm is going to say different solution curves to an ODE cannot cross.

Can sometimes use this

to gain info about the

solution.

Ex: $y' = (y-1)\cos(yt)$

$y(0) = 2.$

Can a solution satisfy

$y(2) = 0$?

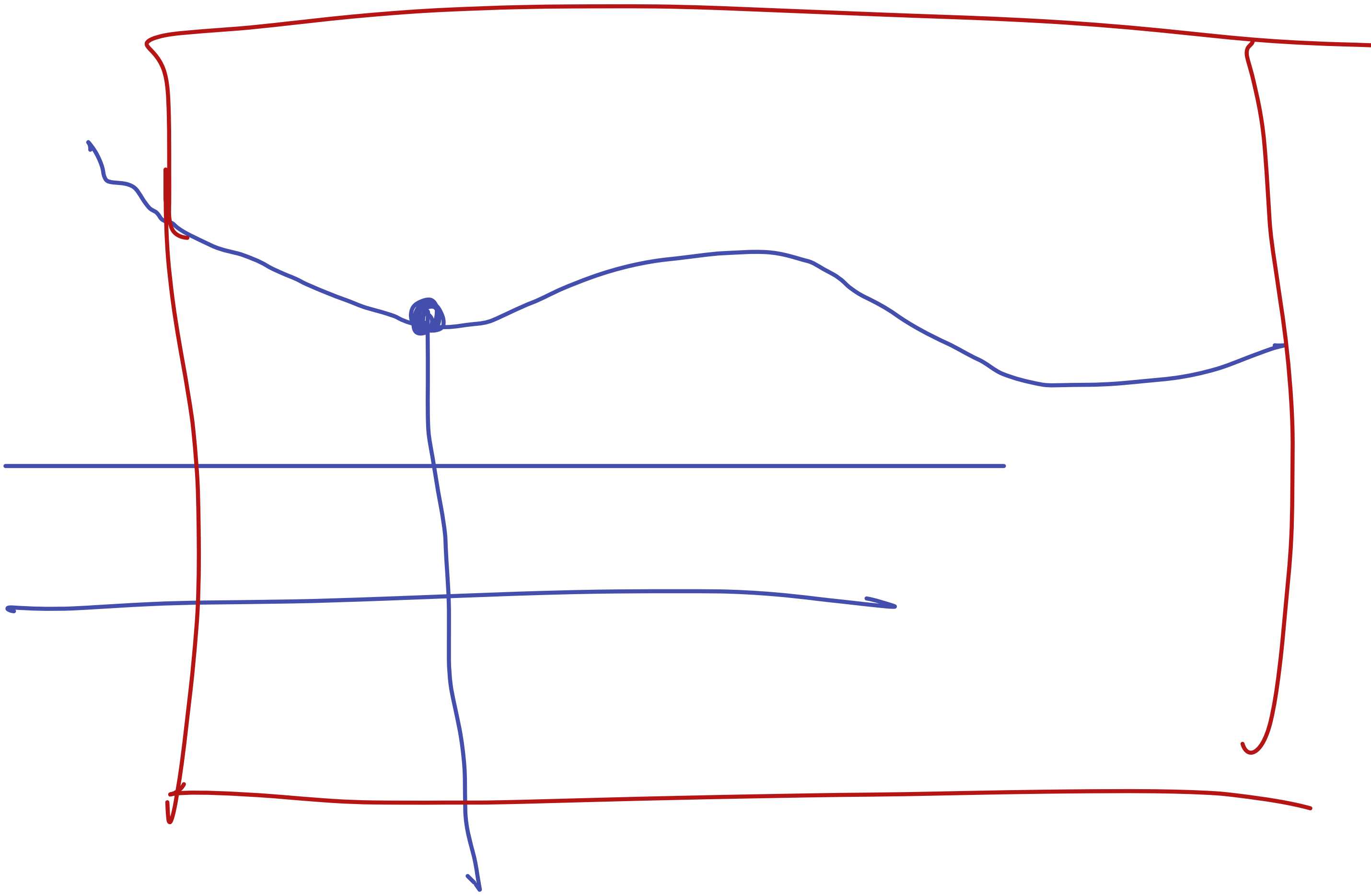
Note that $y(t) = 1$ is

a solution to this IVP.

If $y(2) = 0$ and $y(0) = 2$

then by IVT, we would have $y(t') = 1$ for some $t' \in (0,2)$, but then by uniqueness then we would have to have $y(t) = 1$, which isn't possible.

For any $c \leq 1$, note that the same argument shows $y(t') = c$ for some t' is impossible.



So in particular, we must have $y(t) > 1$ for all t where the sol^n is defined.

Autonomous Equations

$y' = f(y)$ is called <u>autonomous</u>

Auto.

e.g. $y' = y$, $y' = y^2 + \sin(y)$

$y' = y^2 + t$

Not auto.

We're going to analyze autonomous equations qualitatively, since it's going to be easy, and most of the time we can't explicitly solve them.

The slope field will look the same across each horizontal line, so we should expect that a translation of a solution is still a solution.

$y(t)$ a sol^n to $y' = f(y)$.

$y_1(t) = y(t+C)$ for some C,

$y_1'(t) = y'(t+C)$

$$y'(t+c) = f(y(t+c))$$

$$y_1'(t) = f(y_1(t))$$

So indeed, a shift of a sol^n is a sol^n.

Easy solutions to find:

$f(y_0) = 0 \implies$

$y(t) = y_0$ satisfies

$y' = f(y)$, so

Solutions to $f(y) = 0$
correspond to constant solutions
of ODE.

A point y_0 with $f(y_0) = 0$
is called an <u>equilibrium point</u>

and the corresponding solution
$y(t) = y_0$ is called an
<u>equilibrium solution</u>.

Autonomous differential
equations are analyzed

by looking at their equilibrium points.

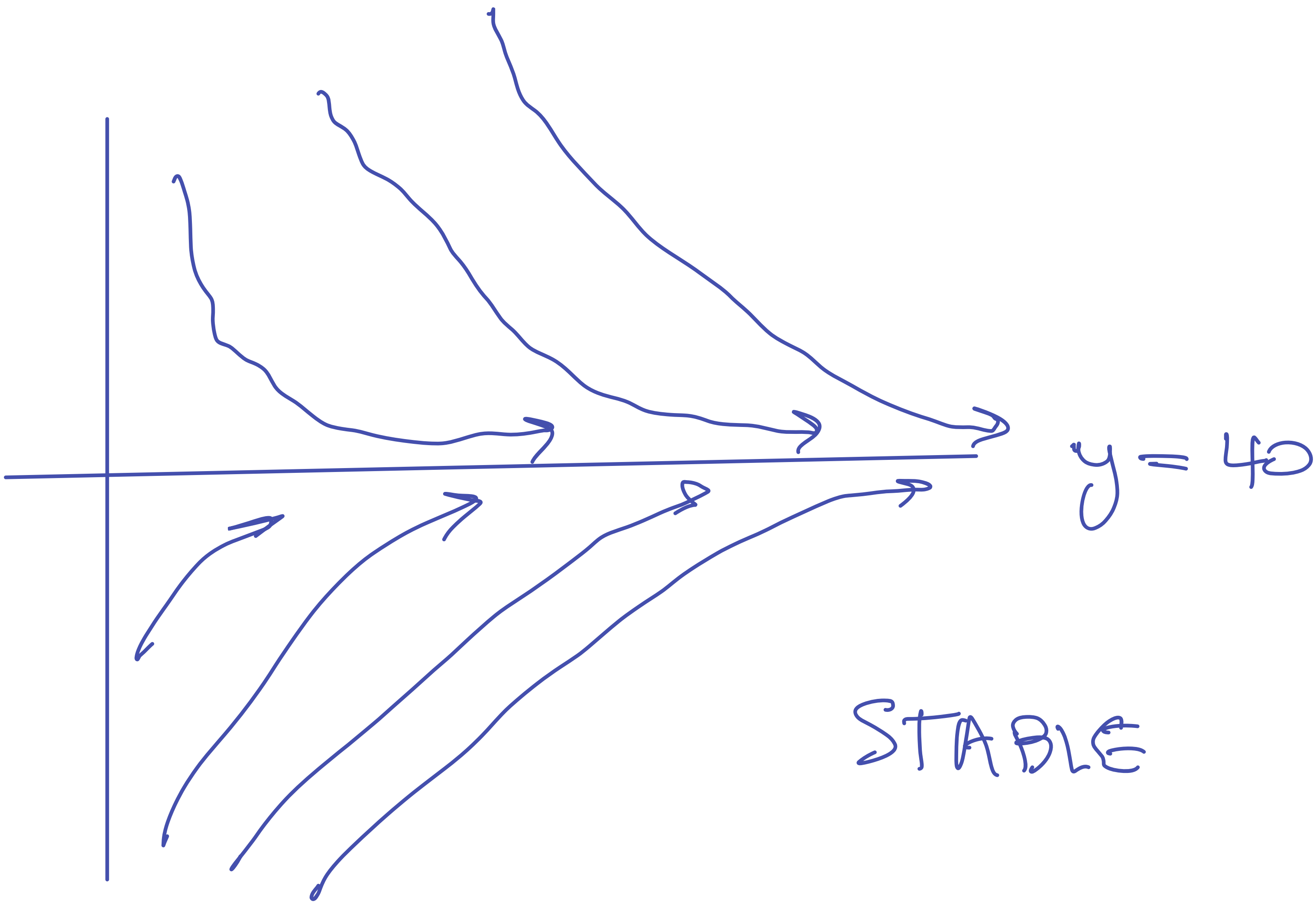
When $f(y)$ and $\frac{df}{dy}$ are both continuous, we can apply existence/uniqueness thm to get unique solutions to IVPs, and since we can't cross the equilibrium sol^{n}s they act as "dividers" for the behavior of sol^{n}s.

An equilibrium point y_0 is called:

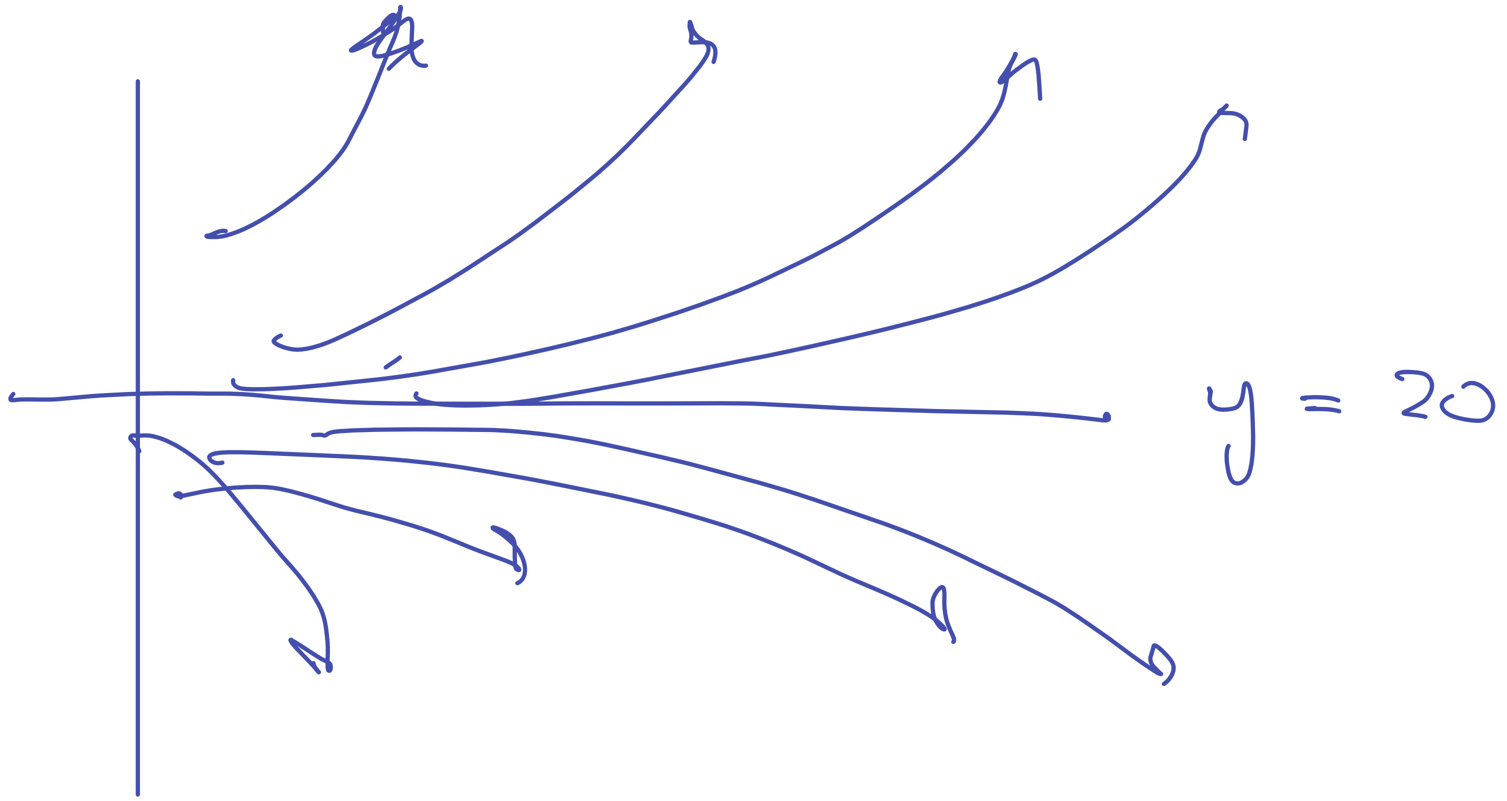
- **Stable** if for any $y(t)$ with initial condition "near" y_0, we have $y \longrightarrow y_0$ as $t \longrightarrow \infty$.

- **Unstable** if for any $y(t)$ with initial condition "near" y_0, y doesn't approach y_0 as $t \longrightarrow \infty$

- <u>semistable</u> if solutions on one side approach and on the other side do not.

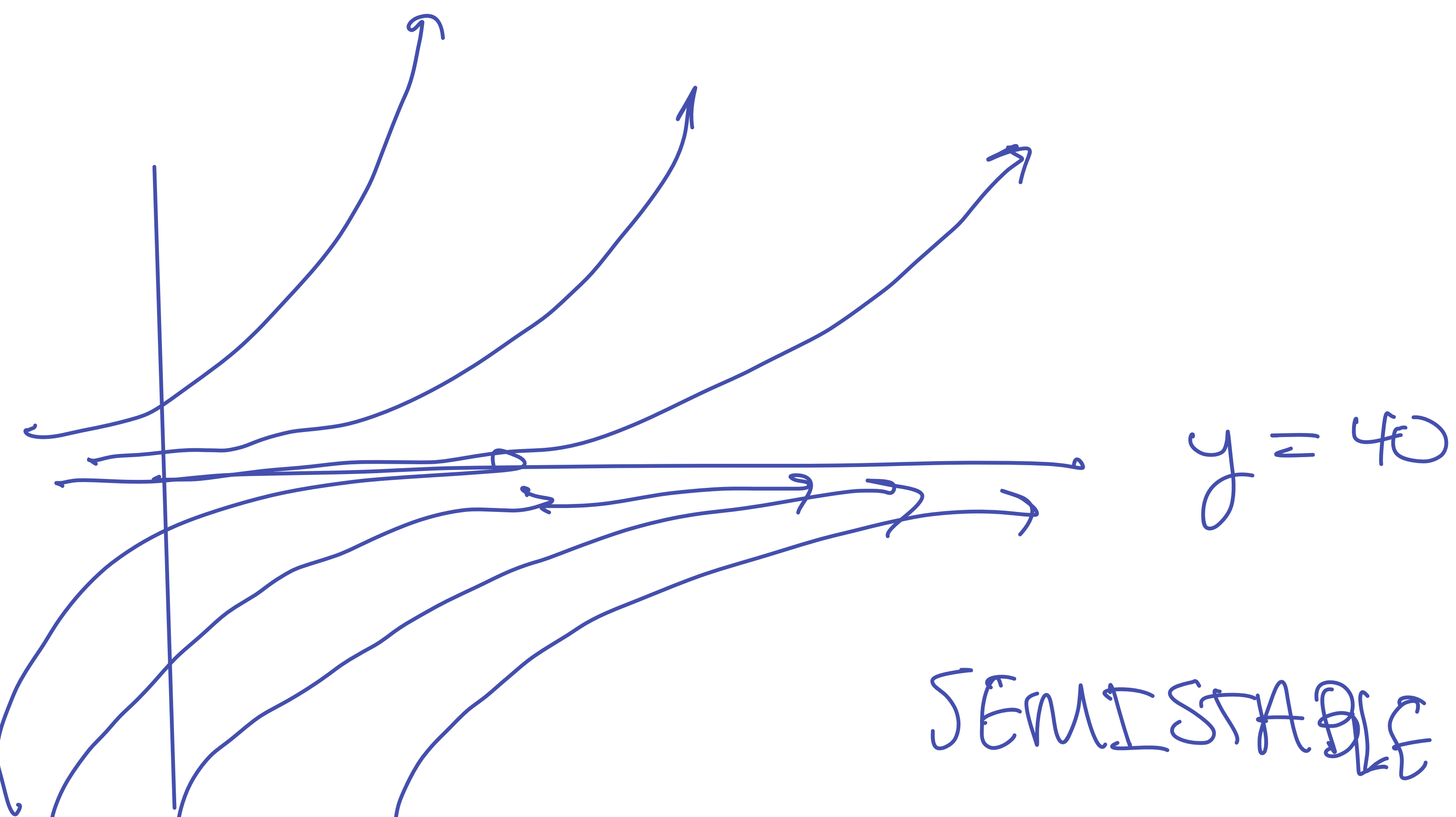


$$y' = -.3(y-40)$$



$$y' = .3(y-20)$$

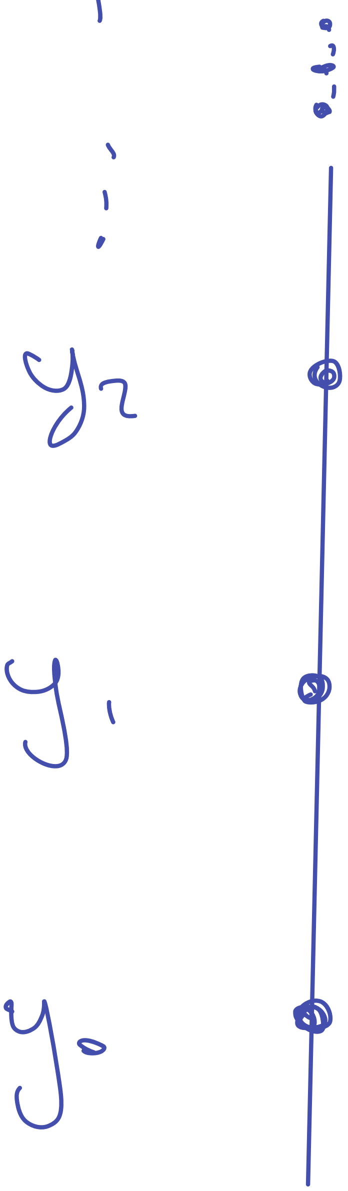
UNSTABLE



SEMISTABLE

$$y' = .05(y-40)^2$$

Stability is classified through the phase line.



$y' = f(y)$ has $y_0, \text{—} \; y_n$ equilibrium sol^ns.

In between two equilibrium pts, $f>0$ or $f<0$, b/c f is cts and equilibrium pts are where $f=0$.

So $y'>0$ or $y'<0$ meaning y is monotonically increasing/decreasing.

- Increasing below + dec. above $\Longrightarrow$ Stable

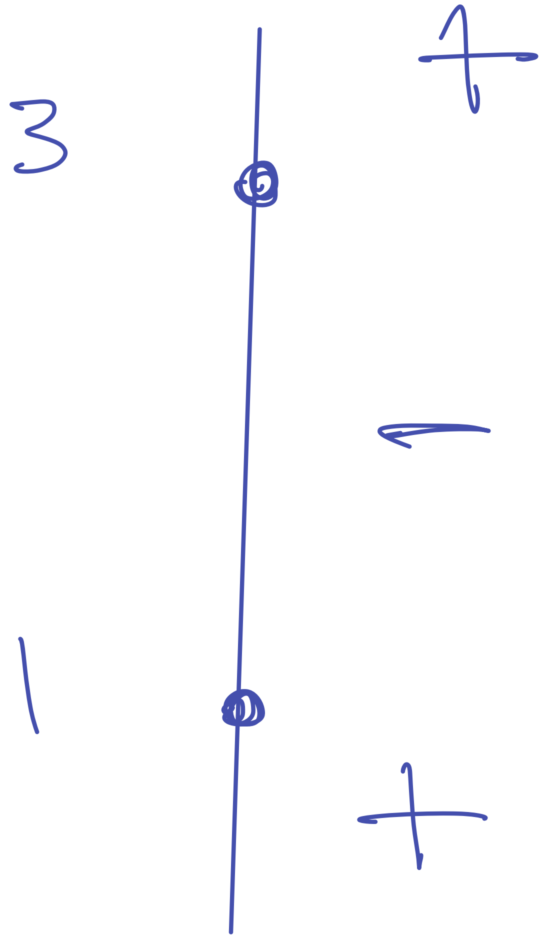
- dec. below + inc. above $\Longrightarrow$ unstable

- dec. below + above or inc. below + above $\Longrightarrow$ Semi-stable.

Ex: $y' = (1-y)(3-y)$

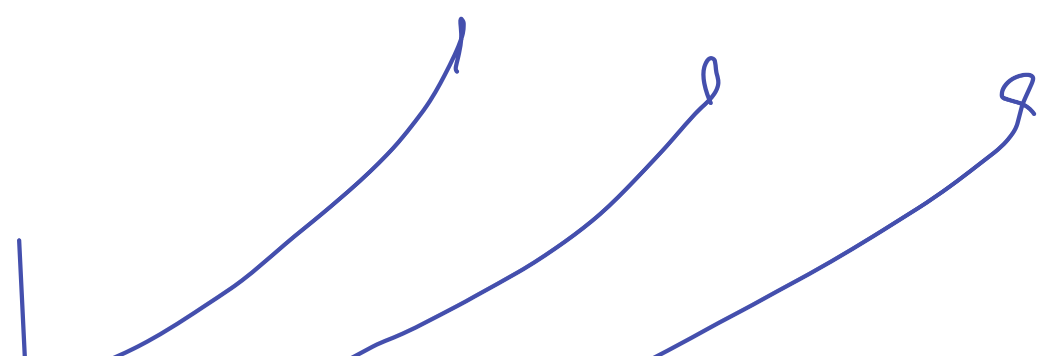
$y_0 = 1, \quad y_1 = 3$

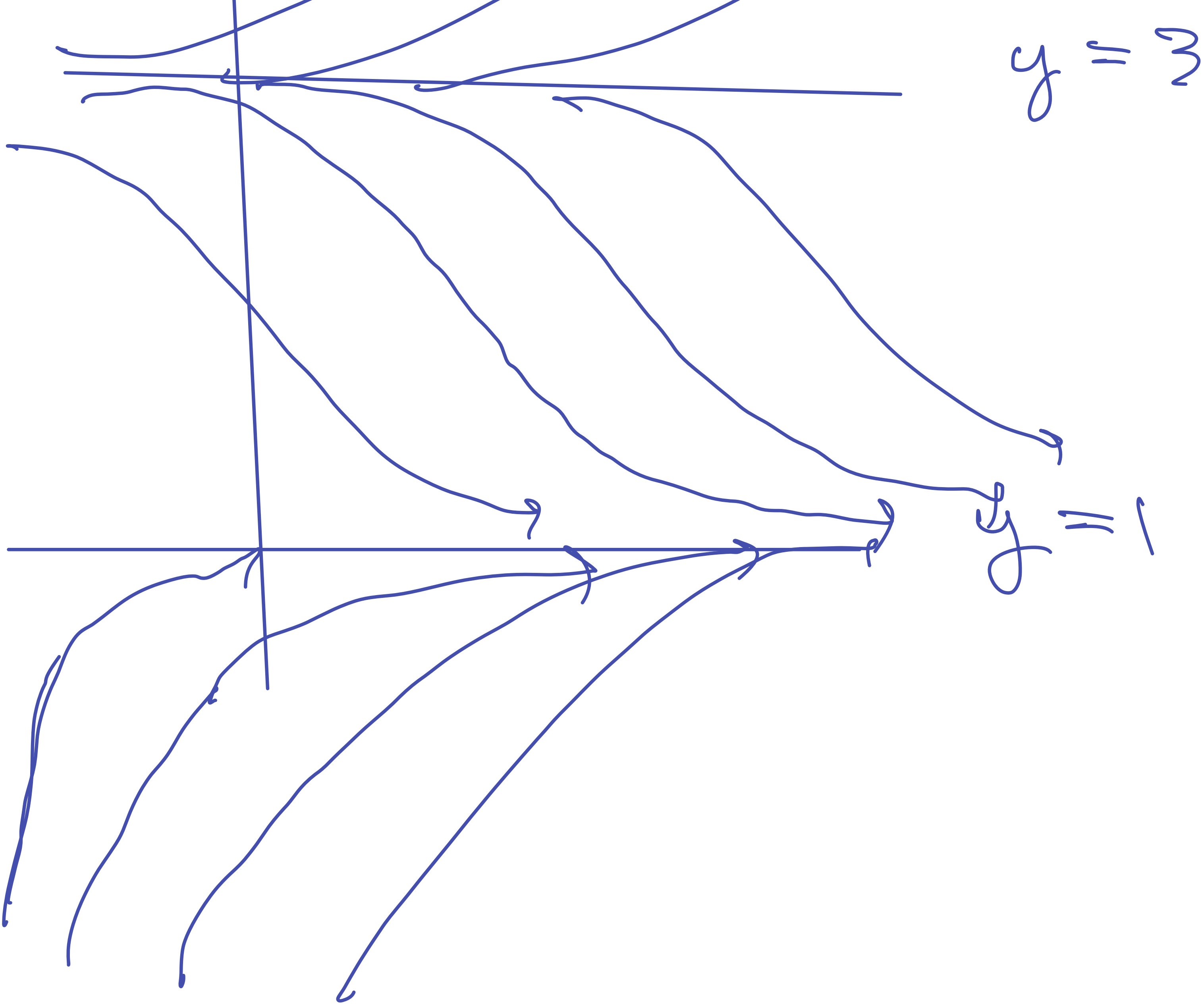
are equilibrium pts.



$y = 3$ unstable

$y = 1$ stable

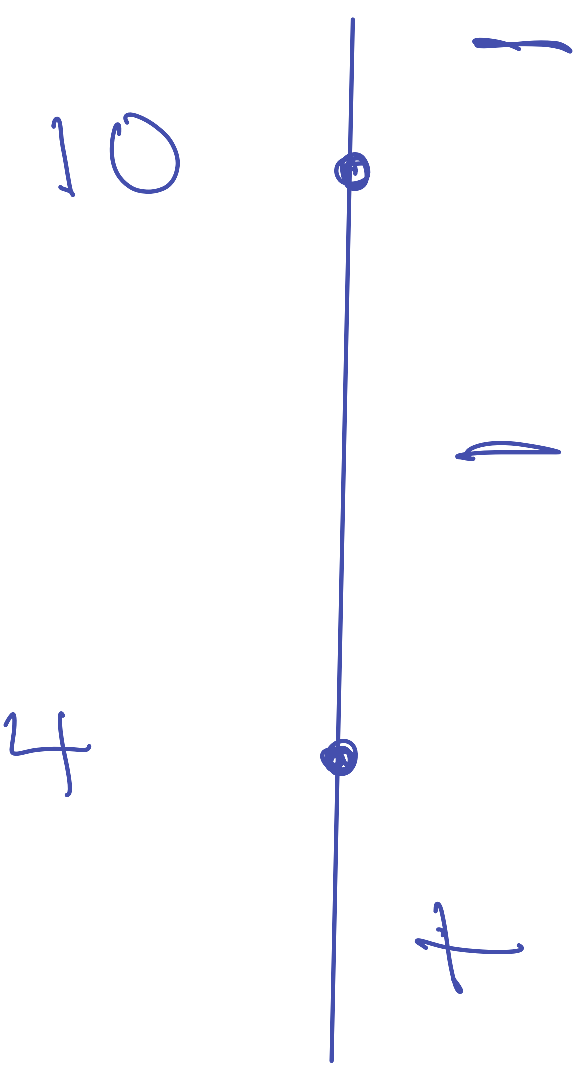




Picture of various solution curves for various initial conditions.

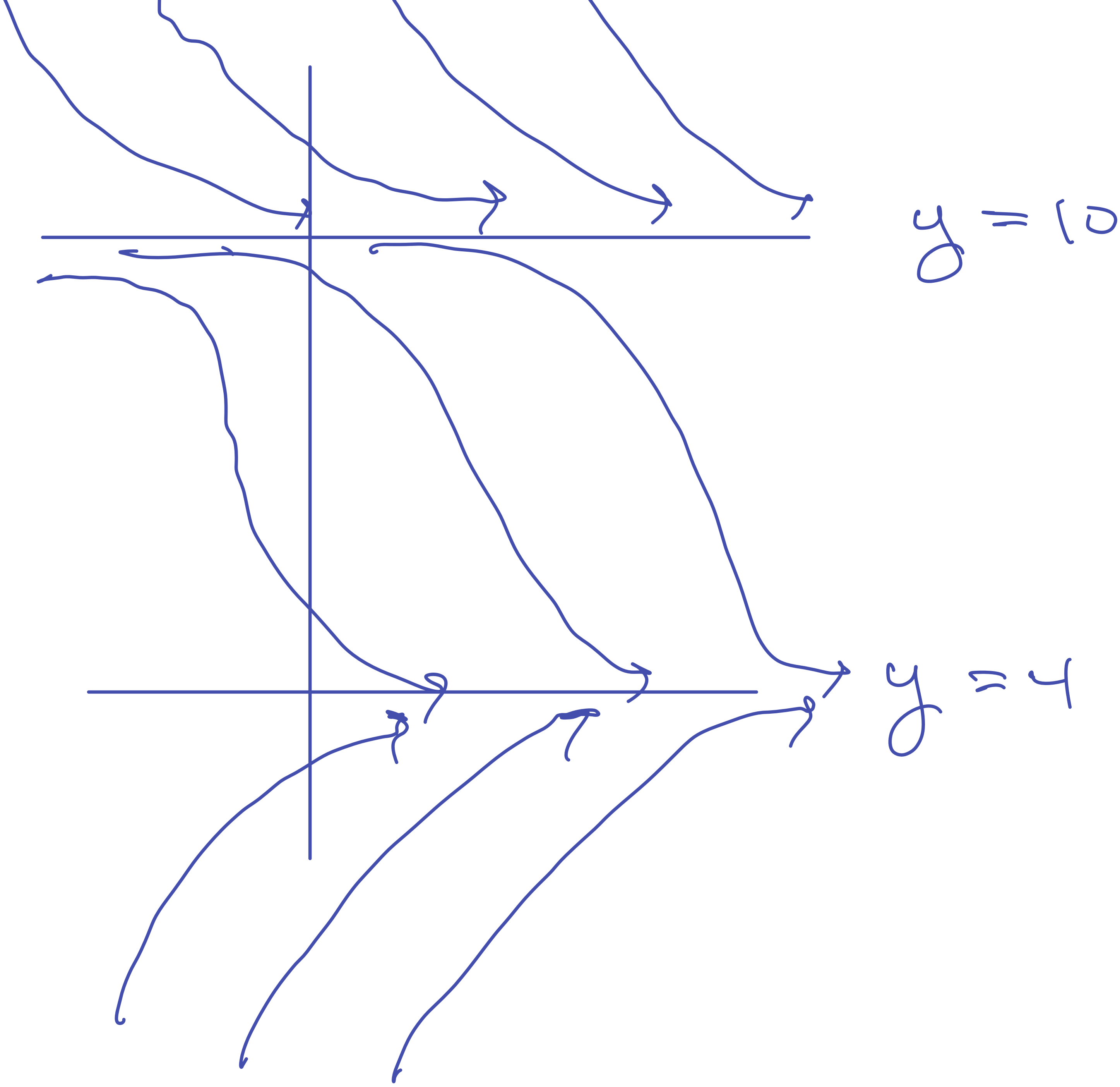
Ex: $y' = -(y-10)^2(y-4)$

$y=4$ and $y=10$ are equilibrium points



$y=4$
Stable

$y=10$
Semi-stable



Stability can also be classified in terms of sign of f'.

y_0 equilibrium pt,

- $f'(y_0) < 0$ then y_0 is stable

- $f'(y_0) > 0$ then y_0 is unstable

- $f'(y_0) = 0$ can't really say anything.

this is generally only relevant when you don't explicitly know y_0.