Exact Differential equations

Goal: $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$

A <u>differential form</u> is an expression of the form

$$\omega = P(x,y)\,dx + Q(x,y)\,dy$$

$$x\,dy + y\,dx = \omega$$

If $y = y(x)$ is a sol^n to the differential equation,

$$dy = y'(x)\,dx$$

So plugging in:

$$P\,dx + Q \cdot \frac{dy}{dx}\,dx = 0$$

$$\omega, \quad \left(P + Q\frac{dy}{dx}\right) dx = 0$$

So saying y is a sol^n to the diff. eq^n is the same thing as saying $\omega = 0$.

Note that the differential equation $y' = f(x,y)$ can be written as

$$\omega = f(x,y)\,dx - dy$$

Often times, when we solve an

ODE the solution is given implicitly.

Ex: $\omega = x\,dx + y\,dy$

This corresponds to

$$\frac{dy}{dx} = -\frac{x}{y}$$

this is separable:

$$\int y\,dy = \int -x\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$x^2 + y^2 = C$$

Any soln y satisfies $x^2+y^2=C$ for some C.

Can explicitly solve for y in this case: $y = \pm\sqrt{c - x^2}$ defined for $|x| \le \sqrt{c}$

<u>def:</u> $F(x,y)$, we define the differential of F, dF as

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy$$

<u>def:</u> A differential form ω is called <u>exact</u> if $\omega = dF$ for some function F.

<u>ex:</u> $x\,dx + y\,dy = \omega$ is exact b/c $\omega = dF$

$$F = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2$$

What does exactness mean for differential equations?

What's going to happen:

$\omega = dF$ sol^s to $\omega = 0$

are going to be given by

$F(x,y) = C$ for some C.

Why?

$\omega = dF$ $F(x,y) = C$

$$\frac{dF}{dx} + \frac{dF}{dy} \cdot \frac{dy}{dx} = 0$$ by chain rule

$\Rightarrow$ $\omega = 0$ by above,

so $F(x,y) = C$ is a sol^ to differential eq^

Ex: $\omega = 2x\,dx + 4y^3\,dy$

exact b/c $\omega = dF$

$F = x^2 + y^4$

this means that $x^2 + y^4 = C$

is the general solution to $\omega = 0$.

How to find F such that $\omega = dF$?

In this example, not too bad:

$$dF = \frac{dF}{dx}dx + \frac{dF}{dy}dy$$

Saying $dF = \omega$ $\Rightarrow$

$$\frac{dF}{dx} = 2x \qquad \frac{dF}{dy} = 4y^3$$

$\Longrightarrow$

$F(x,y) = x^2 + g(y)$ and $F(x,y) = y^4 + h(x)$

From this, we see $x^2 + y^4$ is

One such choice of F that works.

Two questions now:

- Given ω, can we tell if it's exact or not?

- If ω is exact, how can we tell what it comes from?

<u>Thm:</u> $\omega = P\,dx + Q\,dy$ P, Q continuously differentiable

- If ω is exact, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

- If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ holds in some rectangle R, then

ω is exact in R.

Why is this theorem true?

$\omega = dF \qquad P = \frac{\partial F}{\partial x} \qquad Q = \frac{\partial F}{\partial y}$

P, Q diff. $\Rightarrow$ F twice differentiable

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} \qquad \frac{\partial Q}{\partial x} = \frac{\partial^2 F}{\partial y \partial x}$$

mixed partials are equal!

- If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, we want F s.t. $\omega = dF$.

$$\frac{\partial F}{\partial x} = P \Rightarrow F(x,y) = \int P\,dx + g(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int P\,dx + g'(y)$$

$$\Rightarrow \quad \frac{\partial F}{\partial y} = \int \frac{\partial P}{\partial y}\, dx + g'(y)$$

$Q = \frac{\partial F}{\partial y}$ b/c ω exact

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ by assumption

$$\Rightarrow \quad Q = \int \frac{\partial Q}{\partial x}\, dx + g'(y)$$

$$\Rightarrow \quad Q = Q + h(y) + g'(y)$$

$$\Rightarrow \quad 0 = h(y) + g'(y)$$

can solve this differential equation, and take $g(y)$ as any solution.

Ex: $\omega = y^3 dx + 3xy^2 dy$

ω is exact:

$$\frac{dP}{dy} = 3y^2$$ ✓

$$\frac{\partial Q}{dx} = 3y^2$$

To find F, we follow the above procedure:

$$\omega = dF = \frac{dF}{dx} dx + \frac{dF}{dy} dy$$

$$\frac{dF}{dx} = y^3 \implies F(x,y) = xy^3 + g(y)$$

$$\frac{dF}{dy} = 3xy^2 + g'(y)$$

Since ω is exact, $\frac{dF}{dy} = 3xy^2$

$$3xy^2 = 3xy^2 + g'(y)$$

$\Rightarrow \quad g'(y) = 0$

We can take $g(y) = 0$.

So $\omega = dF \qquad F(x,y) = xy^3$.

Solution to $\omega = 0$ are given by $xy^3 = C$.

<u>Ex:</u> $\omega = (6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy$

ω is exact b/c

$\frac{\partial P}{\partial y} = 6x - 3y^2$ ✓

$\frac{\partial Q}{\partial x} = 6x - 3y^2$

To find F,

$$\frac{dF}{dx} = 6xy - y^3 \Longrightarrow$$

$$F(x,y) = 3x^2y - xy^3 + g(y)$$

$$\frac{dF}{dy} = 3x^2 - 3xy^2 + g'(y)$$

ω is exact, $\frac{dF}{dy} = 4y + 3x^2 - 3xy^2$

$$4y + 3x^2 - 3xy^2 = 3x^2 - 3xy^2 + g'(y)$$

$$\Longrightarrow \quad 4y = g'(y)$$

$$\Longrightarrow \quad g(y) = 2y^2$$ is a

choice of sol^n.

$$F(x,y) = 3x^2y - xy^3 + 2y^2$$

$\omega = dF$ and the sol^{n}s is

$\omega = 0$ are

$$3x^2y - xy^3 + 2y^2 = C$$

<u>Rmk:</u> • You could always instead start w/ $\frac{\partial F}{\partial y} = Q$ and integrate first then diff. w.r.t. x.

e.g. 2.6.25 in the textbook

• If you've taken 32B, this procedure should remind you of checking if a vector field is conservative or not and finding a potential function.

- To summarize:

to find F s.t. $\omega = dF$

set $\frac{\partial F}{\partial x} = P$ and integrate

to get $F(x,y) = \int P \, dx + g(y)$.

Differentiate to set up a differential equation so you can solve for $g(y)$.

Integrating factors

we just showed how to solve $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$

when $\omega = P\,dx + Q\,dy$ is exact.

What do we do when ω is not exact?

Theme: fiddle with what we know how to do until it works!

Let's try and find some function $\mu = \mu(x,y)$ s.t.

$\mu\omega$ is exact, so $\mu\omega = dF$.

$$\frac{\partial F}{\partial x} = \mu P \qquad \frac{\partial F}{\partial y} = \mu Q$$

then by doing what we did earlier, the level curves

$$F(x,y) = C \quad \text{still solve}$$

$$\omega = 0.$$

Such a function μ is called an <u>integrating factor</u> for ω.

Unfortunately, finding an integrating factor in general is extremely hard!

Usually, we look for special types of integrating factors related to the specific ODE.

Ex: $\omega = (x+2y^2)\,dx - 2xy\,dy$

then ω is not exact b/c

$$\frac{dP}{dy} = 4y$$

$$\frac{dQ}{dx} = -2y$$

Let's try and find μ so that $\mu\omega$ is exact.

Let's be optimistic, let's look for $\mu = \mu(x)$ that only depends on x.

$$\mu\omega = \mu(x+2y^2)\,dx - 2xy\mu\,dy$$

need:

$$\frac{\partial}{\partial y}\mu(x+2y^2) = \frac{\partial}{\partial x}(-2xy\mu)$$

$$\frac{d}{dy}\mu(x+2y^2) = \frac{\partial}{\partial y}\mu(\cancel{x}+2y^2) + \mu\cdot\frac{\partial}{\partial y}(x+2y^2)$$

$$= 4y\mu$$

$$\frac{d}{dx}(-2xy\mu) = -2y\mu - 2xy\mu'$$

$$4y\mu = -2y\mu - 2xy\mu'$$

$$6y\mu = -2xy\mu'$$

$$-3\mu = x\mu' \qquad \text{this is separable!}$$

$$-\frac{3}{x} = \frac{1}{\mu}\mu'$$

$$\Rightarrow \quad \frac{1}{x^3} = \mu \quad \text{is a choice that solves this.}$$

$$\Rightarrow \quad \mu\omega = \frac{1}{x^3}(x+2y^2)\,dx - \frac{2xy}{x^3}\,dy$$

is going to be exact.

Now need to find F s.t.

$$\mu\omega = dF.$$

$$\mu\omega = \left(\frac{1}{x^2} + \frac{2y^2}{x^3}\right)dx - \frac{2y}{x^2}\,dy$$

$$\frac{dF}{dx} = \frac{1}{x^2} + \frac{2y^2}{x^3} \Rightarrow$$

$$F(x,y) = -\frac{1}{x} - \frac{y^2}{x^2} + g(y)$$

$$\frac{dF}{dy} = -\frac{2y}{x^2} + g'(y)$$

by exactness, also have

$$\frac{dF}{dy} = -\frac{2y}{x^2}$$

$$-\frac{2y}{x^2} = -\frac{2y}{x^2} + g'(y)$$

$$\Rightarrow \quad g'(y) = 0$$

$$\Rightarrow \quad g(y) = 0$$ is a choice that works.

So $F(x,y) = -\frac{1}{x} - \frac{y^2}{x^2}$

and $\mu\omega = dF$, and

So $-\frac{1}{x} - \frac{y^2}{x^2} = C$ are the solutions to $\omega = 0$.

When does $P\,dx + Q\,dy$ have an integrating factor depending only on x?

$$\mu P\,dx + \mu Q\,dy = dF$$

$$\frac{dF}{dx} = \mu P \qquad \frac{dF}{dy} = \mu Q$$

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$$

$$\mu \frac{\partial P}{\partial y} = \mu' Q + \mu \frac{\partial Q}{\partial x}$$

$$\Rightarrow \mu' = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mu$$

if $\boxed{h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}$ depends only on x, then we get a separable equation we can solve.

Similarly, if $\boxed{\frac{1}{P}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}$

depends only on y get an integrating factor depending only on y.

Ex: $(xy-2)\,dx + (x^2 - xy)\,dy$

Not exact.

$$\frac{\partial P}{\partial y} = x$$

$$\frac{\partial Q}{\partial x} = 2x - y$$

Need to find an integrating factor to solve $\omega = 0$.

$$h = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= \frac{1}{x^2 - xy}\left(x - (2x - y)\right) = -\frac{1}{x}$$

$\mu' = -\frac{1}{x}\mu$ has solution

$\mu = 1/x$, so $\mu\omega$ is exact.

$$\mu\omega = \left(y - \frac{2}{x}\right) dx + (x-y)\, dy$$

$$\frac{dF}{dx} = y - \frac{2}{x} \implies$$

$$F(x,y) = xy - 2\ln|x| + g(y)$$

$$\frac{dF}{dy} = x + g'(y)$$

$\mu\omega$ exact, so $\frac{dF}{dy} = x - y$

$$x - y = x + g'(y)$$

$$\implies -y = g'(y)$$

Can take $g(y) = -\frac{1}{2}y^2$

So $F(x,y) = xy - 2\ln|x| - \frac{1}{2}y^2$

works, and $F(x,y) = C$

gives the solutions to $\omega = 0$.

__Homogenous equations:__

$G(x,y)$ is __homogenous of degree n__ if

$$G(tx,ty) = t^n G(x,y)$$

ex: $\frac{1}{x^2+y^2}$, $2x^3 - x^2y + y^3$, $\ln(y/x)$

are all homogenous

$x^2y + y^4$, $\sin(x+y)$ not homogenous.

ω is homogenous if P, Q

homogenous of same degree.

Homogenous differential equations can all be solved by the substitution

$$\boxed{\begin{aligned} y &= xv \\ dy &= v\,dx + x\,dv \end{aligned}}$$

<u>Ex:</u> $\omega = (x^2+y^2)\,dx + xy\,dy$

1) homogenous of degree 2.

$y = vx \qquad dy = x\,dv + v\,dx$

$\omega = (x^2 + v^2x^2)\,dx + x^2v\,(x\,dv + v\,dx)$

$\omega = (x^2 + 2x^2v^2)\,dx + x^3v\,dv$

To solve $\omega = 0$:

$$(1+2v^2)\,dx + xv\,dv = 0$$

$$(1+2v^2)\,dx = -xv\,dv$$

$$-\frac{1}{x}\,dx = \frac{v}{1+2v^2}\,dv$$ which is separable!

$$C - \ln|x| = \frac{1}{4}\ln|1+2v^2|$$

$$\Rightarrow \quad \frac{c}{x} = (1+2v^2)^{1/4}$$

$$\Rightarrow \quad \frac{c}{x^4} = 1+2v^2$$

$$\Rightarrow \quad C = x^4 + 2v^2x^4$$

$$\Rightarrow \quad C = x^4 + 2x^2y^2$$ is sol^n

to $\omega = 0$.