A Geometric look at exact ODEs:

$P = P(x,y)$ $\quad Q = Q(x,y)$ functions

we have the following nice dictionary:

$$P + Q\frac{dy}{dx} = 0 \qquad \text{ODE}$$

$$\updownarrow$$

$$\omega = P\,dx + Q\,dy \qquad \text{Differential Form}$$

$$\updownarrow$$

$$V = (P, Q) \qquad \text{Vector Field.}$$

Start with $P + Q\frac{dy}{dx} = 0$.

Say $y(x)$ is a sol^n to this ODE.

Recall that since $\frac{dy}{dx} = -\frac{P}{Q}$,

the slope field tells us that

at any x_0, that $-\frac{P(x_0,y_0)}{Q(x_0,y_0)}$

is the slope of line tangent to

$y(x)$ at x_0 $\big(y_0 = y(x_0)\big)$.

Hence, the vector field

$(-Q, P)$ consists of vectors

that are tangent to the

trajectory of $y(x)$.

Now, let's say that ω is exact,

so $\omega = dF$ for some F.

$$\Rightarrow \quad P = \frac{dF}{dx} \quad \text{and} \quad Q = \frac{dF}{dy}$$

So $V = (P, Q)$ satisfies

$V = \nabla F$. In other words,

V is orthogonal to level curves of F. (Recall: level curves are $F(x,y) = C$ for various C)

Since $(-Q, P) \cdot (P, Q) = 0$, this means the slope field is orthogonal to V and therefore tangent to the level curves, so that

$y(x)$ must follow a trajectory along some level curve.