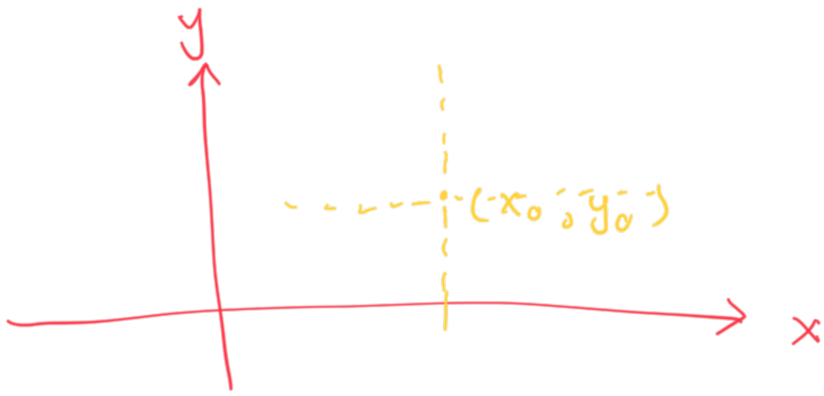
Math 33B : 6/28

Exact Differential Equations

Partial Derivatives

$f(t)$ $\qquad f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$

$f(x,y)$



$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

Examples

1. $z = x^4 y + x y^{-2}$

$$\frac{\partial z}{\partial x} = (4x^3) y + y^{-2} = 4x^3 y + y^{-2}$$

$$\frac{\partial z}{\partial y} = x^4 + x(-2y^{-3}) = x^4 - 2xy^{-3}$$

2. $P = \sin(2s - 3t)$

$$\frac{\partial P}{\partial s} = 2\cos(2s - 3t), \quad \frac{\partial P}{\partial t} = -3\cos(2s - 3t)$$

3. $g(x,y) = \dfrac{xy}{x-y}$, $\dfrac{\partial^2 g}{\partial x \partial y} = ?$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\frac{\partial g}{\partial x}$$

$$\frac{\partial g}{\partial y} = \frac{(x-y)\,x - (xy)(-1)}{(x-y)^2}$$

$$= \frac{x^2}{(x-y)^2}$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{(x-y)^2(2x) - x^2\,(2(x-y))}{(x-y)^4}$$

$$= \frac{2x(x-y)(\;x-y-x)}{(x-y)^4}$$

$$= \frac{-2xy}{(x-y)^3}$$

4. $h(x,y) = \ln(x^3+y^3)$, $h_{xy} = ?$

$$h_{xy} = \frac{\partial^2 h}{\partial x \partial y}$$

$$h_x = \frac{3x^2}{x^3+y^3}$$

$$h_{xy} = 3x^2\left(-\frac{3y^2}{(x^3+y^3)^2}\right) = -\frac{9x^2y^2}{(x^3+y^3)^2}$$

Chain rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)$$

$$f(x,y,z) = f(\,x(s,t),\; y(s,t),\; z(s,t)\,)$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Example

$f(x,y,z) = xy + z^2$, $x = s^2$, $y = 2rs$, $z = r^2$

$$\frac{\partial f}{\partial s} = y\,(2s) + x\,(2r) + (2z)\cdot 0$$

$$= 2ys + 2xr$$

$$= 4rs^2 + 2rs^2$$

$$= 6rs^2$$

$$\frac{\partial f}{\partial r} = y \cdot 0 + x\cdot(2s) + (2z)(2r)$$

$$= 2xs + 4zr$$

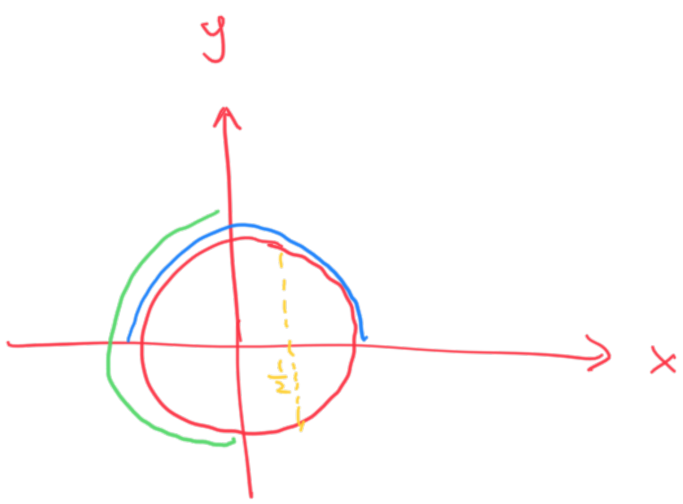
$$= 2s^3 + 4r^3$$

Implicit functions / Implicit Differentiation

$y = f(x)$

$y = \sin x$, $y = e^x + 2x + \sqrt{x-1}$, $y = \ldots$

$F(x,y) = C$



$x^2 + y^2 = 1$

$y = \sqrt{1 - x^2}$

$$X = -\sqrt{1-y^2}$$

Exact Differential Equations

$$P(x,y)dx + Q(x,y)\,dy = 0$$

$$\frac{dy}{dx} = -\frac{P}{Q} \qquad \text{or} \qquad \frac{dx}{dy} = -\frac{Q}{P}$$

$$F(x,y) = C$$

$$F(x, y(x)) = C$$

$$\frac{\partial F}{\partial x}\cdot 1 + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$$\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{P}{Q}$$

$$\frac{\partial F}{\partial x} = \mu P, \quad \frac{\partial F}{\partial y} = \mu Q$$

$$F(x(y), y) = C$$

$$\frac{\partial F}{\partial x}\cdot\frac{dx}{dy} + \frac{\partial F}{\partial y} = 0$$

$$\frac{dx}{dy} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}$$

$$\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}} = \frac{Q}{P}$$

Problem $\quad Pdx + Qdy = 0$

Find $F(x,y)$ and μ such that

$$\begin{cases} \frac{\partial F}{\partial x} = \mu P \\ \frac{\partial F}{\partial y} = \mu Q \end{cases}$$

$F(x,y) = C$ is a solution

Strategy $\quad \dfrac{\partial P}{\partial y} \overset{?}{=} \dfrac{\partial Q}{\partial x}$ (exactness)

$$\frac{\partial F}{\partial x} = P \quad , \quad \frac{\partial F}{\partial y} = Q$$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial P}{\partial y} \quad , \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Examples $\quad -y^2\,dx + x^3\,dy = 0$

$$\frac{\partial}{\partial y}(-y^2) = -2y$$

$$\frac{\partial}{\partial x}(x^3) = 3x^2$$

$$3x^2 \neq -2y$$

(multiply $\frac{1}{x^3y^2}$) $\quad -\frac{1}{x^3}\,dx + \frac{1}{y^2}\,dy = 0$

$$\frac{\partial}{\partial y}\left(-\frac{1}{x^3}\right) = 0 \quad , \quad \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right) = 0$$

$$F(x,y) = \int -\frac{1}{x^3}\,dx + \int \frac{1}{y^2}\,dy$$

Why? $\quad \dfrac{\partial F}{\partial x} = -\dfrac{1}{x^3} \quad , \quad \dfrac{\partial F}{\partial y} = \dfrac{1}{y^2}$

$$F(x,y) = \int -\frac{1}{x^3}\,dx + \int \frac{1}{y^2}\,dy$$

$$= \frac{1}{2x^2} + \left(-\frac{1}{y}\right)$$

$$F(x,y) = C$$

$$\frac{1}{2x^2} - \frac{1}{y} = C$$

$$\frac{1}{2x^2} - C = \frac{1}{y}$$

$$y = \frac{2x^2}{1-2Cx^2}$$