Math 33B : 6/28

Exact Differential Equations

Partial Derivatives

$f(x)$ $\qquad f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$f(x, y)$

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Examples

1. $z = x^4 y + x y^{-2}$

$$\frac{\partial z}{\partial x} = (4x^3) y + y^{-2} = 4x^3 y + y^{-2}$$

$$\frac{\partial z}{\partial y} = x^4 + x(-2) y^{-3} = x^4 - 2xy^{-3}$$

2. $P = \sin(2s - 3t)$

$$\frac{\partial P}{\partial s} = \cos(2s - 3t) \cdot 2 = 2\cos(2s - 3t)$$

$$\frac{\partial P}{\partial t} = \cos(2s - 3t) \cdot (-3) = -3\cos(2s - 3t)$$

3. $g(x,y) = \dfrac{xy}{x-y}$, $\dfrac{\partial^2 g}{\partial x \partial y} = ?$

$$\frac{\partial g}{\partial y} = \frac{(x-y)x - (xy)(-1)}{(x-y)^2} = \frac{x^2}{(x-y)^2}$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x^2}{(x-y)^2}\right)$$

$$= \frac{(x-y)^2(2x) - x^2(2(x-y))}{(x-y)^4}$$

$$= \frac{2x(x-y)(x-y-x)}{(x-y)^4}$$

$$= \frac{-2xy}{(x-y)^3}$$

4. $h(x,y) = \ln(x^3+y^3)$, $h_{xy} = ?$

$$h_x = \frac{3x^2}{x^3+y^3}$$

$$h_{xy} = 3x^2\left(-\frac{3y^2}{(x^3+y^3)^2}\right) = -\frac{9x^2y^2}{(x^3+y^3)^2}$$

Chain rules

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$$

$f(x,y,z)$, $x = x(s,t)$, $y = y(s,t)$, $z = z(s,t)$

$$f(x,y,z) = f(x(s,t), y(s,t), z(s,t))$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}$$

<u>Examples</u>

1. $f(x,y,z) = xy + z^2$, $\quad x = s^2$, $y = 2rs$, $z = r^2$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= y\,(2s) + x\,(2r) + (2z)\cdot 0$$

$$= 2sy + 2rx$$

$$= 2s(2rs) + 2r s^2$$

$$= 6rs^2$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$= y\cdot 0 + x\,(2s) + (2z)(2r)$$

$$= 2xs + 4zr$$

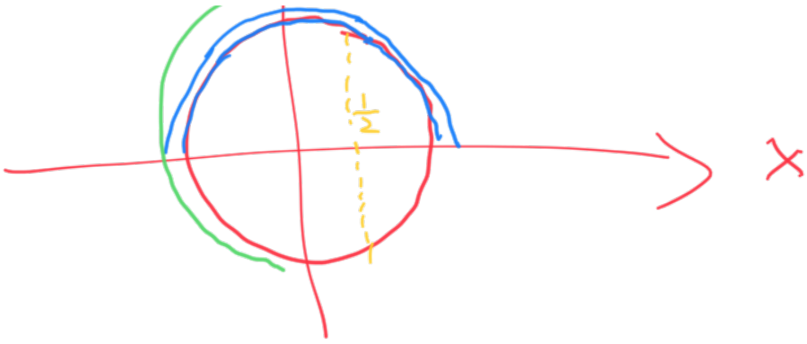
$$= 2s^3 + 4r^3$$

Implicit function / Implicit Differentiation

$$y = f(x)$$

$$F(x,y) = C$$

$$x^2+y^2=1$$



$$y=\sqrt{1-x^2}$$

$$x=-\sqrt{1-y^2}$$

Exact Differential Equations

$$P(x,y)dx+Q(x,y)\,dy=0$$

$$\frac{dy}{dx}=-\frac{P}{Q} \quad \text{or} \quad \frac{dx}{dy}=-\frac{Q}{P}$$

$$F(x,y)=C$$

$$F(x,y(x))=C$$

$$\frac{\partial F}{\partial x}\frac{\partial}{\partial x}(x)+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}=0$$

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$$\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=\frac{P}{Q}$$

$$\frac{\partial F}{\partial x} = \mu P \quad , \quad \frac{\partial F}{\partial y} = \mu Q$$

$$P\,dx + Q\,dy = 0$$

Find $F(x,y)$ and μ

$$\begin{cases} \dfrac{\partial F}{\partial x} = \mu P \\[2ex] \dfrac{\partial F}{\partial y} = \mu Q \end{cases}$$

<u>Examples</u> $\quad -y^2\,dx + x^3\,dy = 0$

(dividing x^3y^2) $-\frac{1}{x^3}\,dx + \frac{1}{y^2}\,dy = 0$

$$F(x,y) = \int -\frac{1}{x^3}\,dx + \int \frac{1}{y^2}\,dy$$

$$= \frac{1}{2x^2} + \left(-\frac{1}{y}\right)$$

$$F(x,y) = C$$

$$\frac{1}{2x^2} - \frac{1}{y} = C$$

$$\frac{1}{2x^2} - C = \frac{1}{y}$$

$$y = \frac{2x^2}{1-2Cx^2}$$

$$P\,dx + Q\,dy = 0$$

$$\frac{\partial P}{\partial y} \stackrel{?}{=} \frac{\partial Q}{\partial x}$$