Linear differential Equations

First order linear ODE:

$y' = a(t)y + b(t)$ for some functions $a(t), b(t)$.

If $b(t) = 0$ we call the equation <u>homogenous</u>

<u>Ex:</u>

$y' = e^t y + t^2$ lin.

$y' = \sin(t)y$ hom. lin.

$y' = \log(y)y - t$ non-linear

linear differential equations are "nice" in the sense that we can explicitly solve them.

Easy case: homogenous equations.

$y' = a(t)y$ this is separable!

$\frac{1}{y}\,dy = a(t)\,dt$

$\ln|y| = \int a(t)\,dt + C$

$|y| = Ce^{\int a(t)\,dt}$

As C is arbitrary, we can recover the correct choice of sign so can drop abs. value:

$$y = Ce^{\int a(t)\,dt}$$

What about the general case?

$$\boxed{y' = a(t)y + b(t)}$$

$$y' - a(t)y = b(t)$$

Going to use a trick to solve the equation: want to multiply the LHS by some function s.t. new LHS = derivative of something.

optimistic: find some $g(t)$ s.t.

$$LHS = \frac{d}{dt} g(t)y$$

$$\underline{g(t)y'} - \underline{g(t)a(t)}y = g(t)b(t)$$

$$\frac{d}{dt} g(t)y = \underline{g'(t)}y + \underline{g(t)}y'$$

for $LHS = \frac{d}{dt} g(t)y$, we need

$$g'(t) = -g(t)\,a(t)$$

$$\frac{g'}{g} = -a$$

$$\Rightarrow \quad \ln|g| = -\int a(t)\,dt + C$$

$$\Rightarrow \quad g = C e^{-\int a(t)\,dt}$$

Let's just take $C=1$ for simplicity.

$$\boxed{g(t) = e^{-\int a(t)\,dt}}$$

What we've deduced is

$$\frac{d}{dt}\left[g(t)\,y\right] = g(t)\,b(t)$$

$\Rightarrow$

$$g(t)\,y = \int g(t)\,b(t)\,dt + C$$

$$\boxed{y(t) = \frac{1}{g(t)}\int g(t)\,b(t)\,dt + \frac{C}{g(t)}}$$

$g(t) = e^{-\int a(t)\,dt}$ is called

an integrating factor

for the differential equation.

Ex: $(t^2+1)\,y' + 3ty = 6t$

$y(1) = 4$

this a linear ODE.

$$y' = -\frac{3t}{t^2+1} y + \frac{6t}{t^2+1}$$

$$g(t) = e^{-\int -\frac{3t}{t^2+1}\, dt}$$

integrating factor

$$\int -\frac{3t}{t^2+1}\, dt$$

$u = t^2+1$

$du = 2t\, dt$

$$= \int -\frac{\frac{3}{2}\, du}{u} = -\frac{3}{2} \ln|u|$$

$$= -\frac{3}{2} \ln(t^2+1)$$

$$g(t) = e^{-\left(-\frac{3}{2}\ln(t^2+1)\right)}$$

$$= (t^2+1)^{3/2}$$

Now multiply by $g(t)$:

$$\frac{d}{dt}\left(y(t^2+1)^{3/2}\right) = 6t(t^2+1)^{1/2}$$

$\Longrightarrow$

$$y(t^2+1)^{3/2} = \int 6t(t^2+1)^{1/2}\,dt$$

$$= 2(t^2+1)^{3/2} + C$$

$$y(t) = 2 + C(t^2+1)^{-3/2}$$

$y(1) = 4$

$$\Longrightarrow \quad 2 + C\, 2^{-3/2} = 4$$

$$2^{-3/2} \cdot C = 2$$

$$C = 2^{5/2}$$

$$\boxed{y(t) = 2 + 2^{5/2}(t^2+1)^{-3/2}}$$

Another way of thinking about first order linear ODEs:

$$\boxed{y' = a(t)y + b(t)}$$

Start with homogenous eqn:

$$y_h' = a(t)y_h$$

this we know how to solve:

$$y_h = Ce^{\int a(t)\,dt}$$

(take $C=1$ for simplicity)

$$\boxed{y_h = e^{\int a(t)\,dt}}$$

$$y = v y_h \qquad v = y/y_h$$

let's figure out what v

Should be to get a sol?

$$(vy_h)' = v'y_h + vy_h'$$

$$v'y_h + vy_h' = a(t)vy_h + b(t)$$

$$v'y_h + a(t)vy_h = a(t)vy_h + b(t)$$

$$\Rightarrow \quad v' = b(t)/y_h$$

$$\boxed{v = \int \frac{b(t)}{y_h(t)}\,dt + C}$$

and $\boxed{y = vy_h}$ is a sol?

This technique is called <u>variation of parameter</u>

Key idea: try modifying things you <u>do</u> know how to solve to solve new things.

Ex: $$y' + y\cot(t) = \cos(t)$$

homogeneous equation:

$$y' + y\cot(t) = 0$$

$$y' = -y\cot(t)$$

$$\int \frac{1}{y}\,dy = \int -\cot(t)\,dt$$

$$\ln|y| = -\ln|\sin(t)| + C$$

$$y = C\csc(t)$$

let's take $C = 1$:

$$y_h = \csc(t)$$

Now need the fudge factor:

$$v' = \frac{b(t)}{y_h} = \frac{\cos(t)}{\csc(t)} = \cos(t)\sin(t)$$

$$\Rightarrow \quad v(t) = \frac{1}{2}\sin^2(t) + C$$

$$y = v(t)\,y_h(t)$$

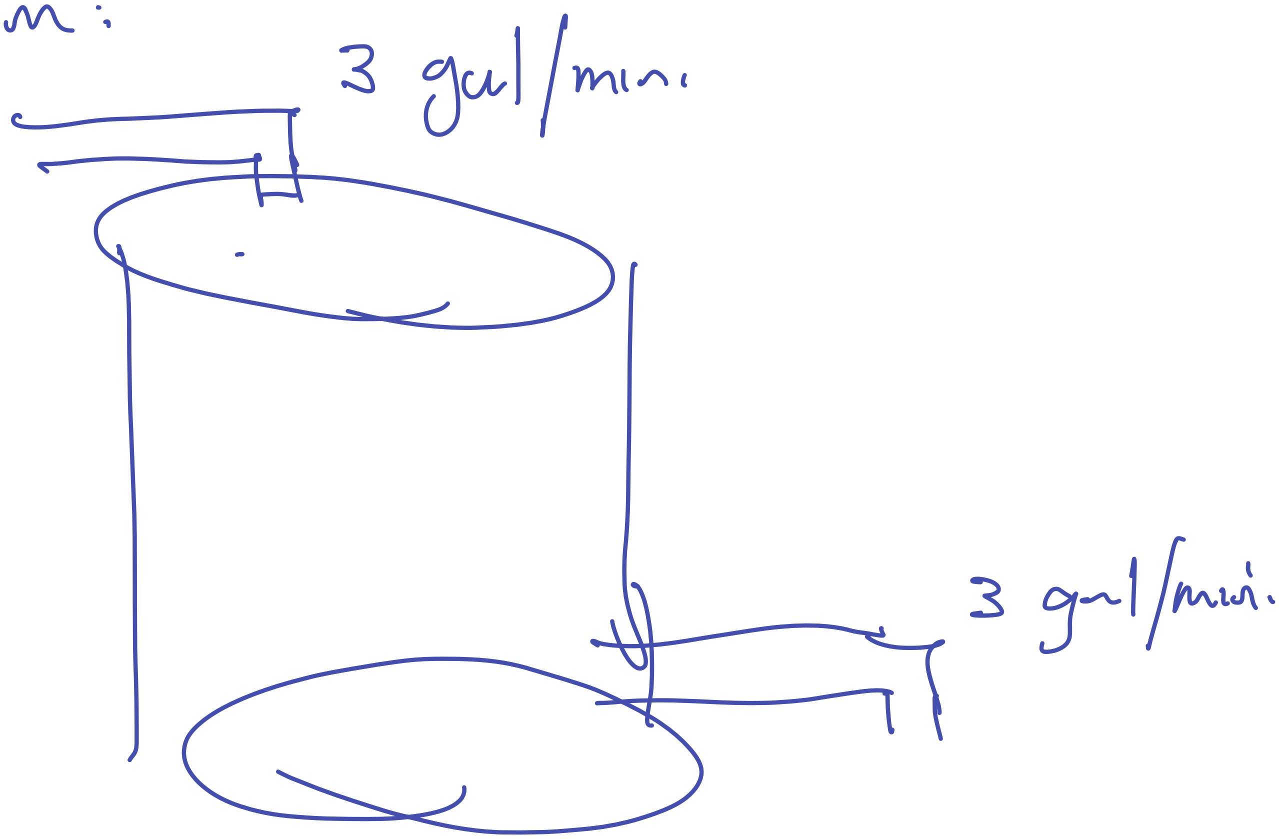
$$= \left(\frac{1}{2}\sin^2(t) + C\right)\csc(t)$$

$$\boxed{y = \frac{1}{2}\sin(t) + C\csc(t)}$$

Some applications

Application #1: Mixing problems

Problem:



Suppose we have a tank with 100 gal of water in it. At time $t=0$, a solution of 2 lb salt/gal enters the tank at a rate of 3 gal/min.

Let's suppose water leaves at 3 gal/min to keep the volume of the tank constant at 100 gal.

At any given time t, how much salt is in the tank?

$y(t)$ = # lbs of salt in the tank after t minutes.

Our goal: find a formula for $y(t)$. Will set up an ODE and solve it explicitly.

$y'(t)$ = rate of change of salt in tank w.r.t. time.

rate of change = rate in − rate out

rate in: this one is easy!

2 lb salt/gal $\cdot$ 3 gal/min

= 6 lb salt/min.

rate out is slightly harder.

We assume that mixing happens instantly, so that concentration of salt doesn't depend on location in the tank.

rate out:

$$3\ \text{gal/min} \cdot \underbrace{y(t)/100}_{\text{concentration of salt lbs/gal}}$$

$= 3y(t)/100$ lbs/min

rate of change:

$$y'(t) = 6 - 3y(t)/100$$

$$y' = 6 - \frac{3y}{100}$$

$$y(0) = 0$$

this gives us IVP to solve.

$$y' + \frac{3}{100}y = 6$$

$$y_h' + \frac{3}{100}y_h = 0$$ homogeneous eqn

$$y_h = e^{-3/100\,t}$$

$$v' = \frac{6}{y_h} = 6e^{3/100\,t}$$

$$v = 200e^{3/100\,t} + C$$

$$y(t) = \left(200e^{3/100\,t} + C\right)e^{-3/100\,t}$$

$$= 200 + Ce^{-3/100\,t}$$

$y(0) = 0$:

$$0 = 200 + C$$

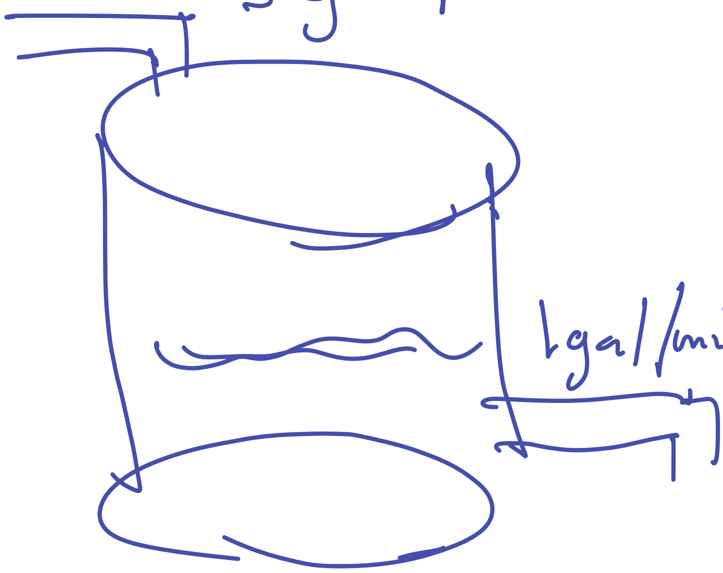
$$\Rightarrow C = -200.$$

$$\boxed{y(t) = 200 - 200e^{-3/100\,t}}$$

Ex: Let's modify the setup a bit.

3 gal/min

600 gal tank



300 gal of water initially.

1.5 lbs salt/gal coming in @ 3 gal/min. Water leaving at a rate of 1 gal/min.

At any given time, how much salt is in the tank?

$y(t)$ = # lbs of salt at time t.

rate of change = rate in - rate out

again, rate in is easy:

$$3 \text{ gal/min} \cdot 1.5 \text{ lbs salt/gal}$$
$$= 4.5 \text{ lbs salt/min.}$$

rate out:

Same as before:

$$1 \text{ gal/min} \cdot \text{concentration at time } t.$$

Concentration: $\dfrac{y(t)}{\text{Volume}(t)} = \dfrac{y(t)}{300+2t}$

water increases at 2 gal/min

$\Rightarrow$

$$\text{Volume}(t) = 300 + 2t$$

$$\Rightarrow \text{rate out} = \frac{y(t)}{300+2t} \; lb/min$$

$$y'(t) = 4.5 - \frac{y(t)}{300+2t}$$

$$y(0) = 0$$

$$y' = 4.5 - \frac{1}{300+2t} y$$

First order linear ODE, solve using integrating factor / variation of parameters.

Integrating factor:

$$g(t) = e^{-\int -1/300+2t \, dt}$$

$$= e^{\frac{1}{2}\ln|300+2t|} = \sqrt{300+2t}$$

$$\Rightarrow \frac{d}{dt}\left[\sqrt{300+2t}\, y\right] = 4.5\sqrt{300+2t}$$

$$\Rightarrow \sqrt{300+2t}\, y = \int 4.5\sqrt{300+2t}\, dt$$

$$= \frac{3}{2}(300+2t)^{3/2} + C$$

$$\Rightarrow y = \frac{3}{2}(300+2t) + \frac{C}{\sqrt{300+2t}}$$

$$y(t) = 450 + 3t + \frac{C}{\sqrt{300+2t}}$$

$$y(0) = 0$$

$$\Rightarrow C = -4500\sqrt{3}$$

$$y(t) = 450 + 3t - \frac{4500\sqrt{3}}{\sqrt{300+2t}}$$

How much salt is in the tank when the tank is full?

tank full when Volume$(t) = 600$

i.e. $t = 150$

$y(150) \simeq 582$ lbs.

Application #2 : Newton's law of cooling.

$T(t)$ = temperature of some object at time t

A = temperature of ambient space (can vary w/ time, but assume constant)

Law of cooling: K some constant that depends on the object

$$T' = -K(T - A)$$

this is separable:

$$\frac{dT}{T-A} = -K\,dt$$

Solving yields

$$T = A + Ce^{-Kt}$$

Ex: bottle of champagne @ 25°C is placed into a 3°C fridge.

After 1/2 hr, the bottle is a 15°C.

How long until it hits 4°C?

$$T(t) = 3 + Ce^{-kt}$$

$$T(0) = 25 \implies 25 = 3 + C$$

$$C = 22.$$

$$T(t) = 3 + 22e^{-kt}$$

$$T(30) = 15 = 3 + 22e^{-30k}$$

$$k = -\ln(12/22)/30 \approx .02$$

$$T(t) = 3 + 22e^{0.02t}$$

$$4 = 3 + 22e^{-.02t}$$

$$t = \ln(1/22)/-.02 \approx 155$$

roughly 2hr 35 mins.