Math 33 B : 6/23

1st order linear differential equations

$$\underline{y'} + a(t)\,\underline{y} = \underline{b(t)}$$

Integrating factor

$a(t) = 0$

$$y' = b(t)$$

$$\int y'\,dt = \int b(t)\,dt$$

$$y(t) = \int b(t)\,dt$$

$$y' + ay = b$$

Let $\mu(t) = e^{\int a(t)\,dt}$

$$\underline{\mu y' + a\mu y} = b\mu$$

$$\underline{(\mu y)' = b\mu}$$

Why?

$$(\mu y)' = \mu y' + \mu' y$$

$$= \mu y' + \left(e^{\int a(t)\,dt}\right)' y$$

$$= \mu y' + e^{\int a(t)\,dt}\,a(t)\,y$$

$$= \mu y' + a\mu y$$

$$\mu y = \int b(t)\,\mu(t)\,dt$$

$$y = e^{-\int a(t)\,dt} \int b(t)\,\mu(t)\,dt$$

Example 1 $\quad y' + (\cos t)y = 0 \quad$ (1st order linear)

Let $\mu(t) = e^{\int \cos t\,dt} = e^{\sin t}$

$$e^{\sin t}\left(y' + (\cos t)y\right) = 0$$

$$\left(e^{\sin t} \cdot y\right)' = 0$$

$$e^{\sin t} y = C$$

$$y(t) = Ce^{-\sin t}$$

Alternatively, $\quad y' + \cos t\, y = 0 \quad$ (separable equations)

$$\frac{dy}{dt} + \cos t\, y = 0$$

$$\frac{1}{y}\,dt = -\cos t\,dt$$

$$\int \frac{1}{y}\,dt = \int -\cos t\,dt$$

$$\ln|y| = -\sin t + C$$

$$|y| = e^{-\sin t} \cdot e^{C}$$

$$y = C_1 e^{-\sin t}$$

Example 2 $\quad \{ \; ty' + 2y = \frac{\sin t}{t}$

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$t y' + 2y = \frac{\sin t}{t}$$

$$y' + \frac{2}{t} y = \frac{\sin t}{t^2}$$

Let $\mu(t) = e^{\int \frac{2}{t} dt} = e^{2\ln|t|} = |t|^2 = t^2$

$$t^2\left(y' + \frac{2}{t} y\right) = \sin t$$

$$(t^2 y)' = \sin t$$

$$t^2 y = \int \sin t \, dt$$

$$t^2 y = -\cos t + C$$

$$y = \frac{-\cos t + C}{t^2}$$

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$\frac{-\cos\left(\frac{\pi}{2}\right) + C}{\left(\frac{\pi}{2}\right)^2} = \frac{2}{\pi}$$

$$C = \frac{2}{\pi} \cdot \left(\frac{\pi}{2}\right)^2$$

$$C = \frac{\pi}{2}$$

$$y(t) = \frac{(-\cos t) + \frac{\pi}{2}}{t^2}$$

Example 3

$$y' = \frac{ty - t}{t^2 - 3t + 2}$$

$$y' = \frac{t(y-1)}{t^2-3t+2}$$

$$\frac{1}{y-1}\,dy = \frac{t}{t^2-3t+2}\,dt$$

$$\int \frac{1}{y-1}\,dy = \int \frac{t}{t^2-3t+2}\,dt$$

$$\ln|y-1| = \int \frac{t}{t^2-3t+2}\,dt$$

$$\int \frac{t}{t^2-3t+2}\,dt = ?$$

partial fraction decomposition

$$\frac{t}{t^2-3t+2} = \frac{t}{(t-2)\cdot(t-1)} = \frac{A}{t-2} + \frac{B}{t-1}$$

$$t = A(t-1) + B(t-2)$$

Let $t=1$, then

$$1 = A(1-1) + B(1-2)$$

$$B = -1$$

Let $t=2$, then

$$2 = A(2-1)$$

$$A = 2$$

$$\frac{t}{t^2-3t+2} = \frac{2}{t-2} - \frac{1}{t-1}$$

$$\ln|y-1| = \int \frac{t}{t^2-3t+2}\,dt$$

$$\ln|y-1| = \int \frac{2}{t-2}\,dt - \int \frac{1}{t-1}\,dt$$

$$\ln|y-1| = 2\ln|t-2| - \ln|t-1| + C$$

$$|y-1| = e^{2\ln|t-2|}\, e^{-\ln|t-1|}\, e^{C}$$

$$|y-1| = |t-2|^2\, |t-1|^{-1}\, e^{c}$$

$$|y-1| = C_1 \frac{|t-2|^2}{|t-1|}$$

$$y = 1 + C_2 \frac{(t-2)^2}{\underline{t-1}}$$