Differential Equations

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{ Tuesday 2pm - 3pm
{ Tuesday 4:30pm - 5:30pm

Review of Calculus

$$\frac{dy}{dt} = y$$

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

Example

1. $\frac{d}{dx} x^n = n \cdot x^{n-1}$

$\frac{d}{dx} x^2 = 2x$, $\frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}}$

2. $\frac{d}{dx} e^x = e^x$

3. $\frac{d}{dx} \ln x = \frac{1}{x}$

4. $\frac{d}{dx} \sin x = \cos x$

5. $\frac{d}{dx}\cos x = -\sin x$

$$(f+g)' = f' + g'$$

$$(cf)' = c f'$$

product rule

$$(fg)' = f'g + f\cdot g' \quad , \quad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

chain rule

$$f(g(x))' = f'(g(x))\, g'(x)$$

Example $f(t) = \exp(t)\exp(\exp(t))$

$$f'(t) = \left(\frac{d}{dt}\exp(t)\right)\exp(\exp(t)) + \exp(t)\frac{d}{dt}(\exp(\exp(t)))$$

$$= \exp(t)\exp(\exp(t)) + \exp(t)\exp(\exp(t))\exp(t)$$

$$= (\exp(t)+1)\exp(t)\exp(\exp(t))$$

Inverse functions

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

Example

$\frac{d}{dx}\arctan x = ??$

$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$

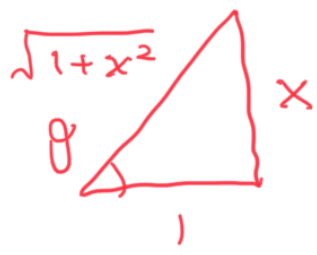
$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$= \cos x \frac{1}{\cos x} - \frac{1}{\cos^2 x}\cdot(-\sin x)\sin x$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

$$\frac{d}{dx}\arctan x = \frac{1}{\sec^2(\arctan x)}$$



$$= \frac{1}{1+x^2}$$

Examples

(1) $f(t) = \sin(\cos(t)) + \arctan(4t)$

$$f'(t) = \cos(\cos(t))\,(-\sin t) + \frac{1}{1+(4t)^2}\cdot 4$$

$$= -\sin t \cos(\cos t) + \frac{4}{1+16t^2}$$

(2) $f(t) = t^2 + 2t^{\frac{3}{2}} + 3t^{\frac{4}{3}} + 4t^{\frac{5}{4}}$

$$f'(t) = 2t + 2\cdot\frac{3}{2}t^{\frac{1}{2}} + 3\cdot\frac{4}{3}t^{\frac{1}{3}} + 4\cdot\frac{5}{4}t^{\frac{1}{4}}$$

$$= 2t + 3t^{\frac{1}{2}} + 4t^{\frac{1}{3}} + 5t^{\frac{1}{4}}$$

Integration

Fundamental theorem of Calculus

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

$$\int f'(t)\,dt = f(t) + C$$

Examples

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ when $n \neq -1$

2. $\int \frac{1}{x}\,dx = \ln|x| + C$

3. $\int e^x\,dx = e^x + C$

4. $\int \sin x\,dx = -\cos x + C$

5. $\int \cos x\,dx = \sin x + C$

Change of variables:

$$\int f(x)\,dx = \int f(g(y))\,g'(y)\,dy$$

Examples

1. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$= \int -\frac{1}{u}\,du$

$= -\ln|u| + C = -\ln|\cos x| + C$

$= \ln|\cos x|^{-1} + C$

$= \ln|\sec x| + C$ $= \ln|\sec x| + C$

2. $\int \sin^3 x \cos x \, dx$ $\quad u = \sin x$

$= \int u^3 \, du$ $\quad du = \cos x \, dx$

$= \frac{1}{4} u^4 + C$

$= \frac{1}{4} (\sin x)^4 + C$

Integration by part:

$$\int u \, dv = uv - \int v \, du$$

$\int x e^x \, dx = x e^x - \int e^x \, dx$ $\quad u = x \quad v = e^x$

$du = dx \quad dv = e^x dx$

$= x e^x - e^x + C$

$\int x^2 e^x \, dx = x^2 e^x - \int 2x e^x \, dx$ $\quad x^2 \quad e^x$

$2x \quad e^x$

$= x^2 e^x - 2(x e^x - e^x) + C$

$= x^2 e^x - 2x e^x + 2e^x + C$

$\int x^n e^x \, dx =$

$$y'(t) = t \quad , \quad y(1) = 2$$

$$y'(t) = t$$

$$\int y'(t)\, dt = \int t\, dt$$

$$y(t) = \frac{1}{2} \cdot t^2 + C$$

$$y(1) = \frac{1}{2} \cdot 1^2 + C = 2$$

$$\Rightarrow \quad C = \frac{3}{2}$$

$$y(t) = \frac{t^2}{2} + \frac{3}{2}$$