Math 33 B 6/23

1st order linear differential equations

$$y' + a(t)\,y = b(t)$$

Integrating factor

$a(t) = 0$ $\qquad y' = b(t)$

$$\int y'\,dt = \int b(t)\,dt$$

$$y(t) = \int b(t)\,dt$$

$$y' + a\,y = b$$

Let $\mu(t) = e^{\int a(t)\,dt}$

$$\mu y' + a\mu y = \mu b$$

$$(\mu y)' = \mu b$$

Why? $\quad (\mu y)' = \mu y' + \mu' y$

$$= \mu y' + \left(e^{\int a(t)\,dt}\right)' y$$

$$= \mu y' + e^{\int a(t)\,dt}\, a(t)\, y$$

$$= \mu y' + a\mu y$$

$$\mu y = \int \mu(t)\, b(t)\,dt$$

$$y = \frac{1}{\mu(t)} \int \mu(t)\, b(t)\,dt$$

Examples

$$y' + \cos t\, y = 0 \qquad \text{(1st order linear)}$$

Let $\mu(t) = e^{\int \cos t\, dt} = e^{\sin t}$

$$e^{\sin t}(y' + \cos t\, y) = 0$$

$$(e^{\sin t} y)' = 0$$

$$e^{\sin t} y = C$$

$$y = C e^{-\sin t}$$

Alternatively, $y' + \cos t\, y = 0$ (separable equations)

$$\frac{dy}{dt} = -(\cos t) y$$

$$\frac{1}{y}\, dy = -\cos t\, dt$$

$$\int \frac{1}{y}\, dy = \int -\cos t\, dt$$

$$\ln |y| = -\sin t + C_0$$

$$|y| = e^{C_0} e^{-\sin t}$$

$$y = C_1 e^{-\sin t}$$

$$y' + a(t) y = 0$$

$$\frac{dy}{dt} + a(t)\, y = 0$$

$$\frac{1}{y}\, dy = -a(t)\, dt$$

$$\ln|y| = \int -a(t)\, dt + C$$

$$y = C_1 e^{\int -a(t)\, dt}$$

Example 2

$$\begin{cases} t\, y' + 2y = \dfrac{\sin t}{t} \\ y\left(\dfrac{\pi}{2}\right) = \dfrac{2}{\pi} \end{cases}$$

$$t\, y' + 2y = \frac{\sin t}{t}$$

$$y' + \frac{2}{t}\, y = \frac{\sin t}{t^2}$$

Let $u(t) = e^{\int \frac{2}{t}\, dt} = e^{2\ln|t|} = |t|^2 = t^2$

$$t^2\left(y' + \frac{2}{t}\, y\right) = \sin t$$

$$(t^2 y)' = \sin t$$

$$t^2 y = -\cos t + C$$

$$y(t) = \frac{-\cos t + C}{t^2}$$

$$y\left(\frac{\pi}{2}\right) = \frac{2}{\pi}$$

$$\frac{-\cos\frac{\pi}{2} + C}{\left(\frac{\pi}{2}\right)^2} = \frac{2}{\pi}$$

$$C = \left(\frac{\pi}{2}\right)^2 \cdot \frac{2}{\pi} = \frac{\pi}{2}$$

$$y(t) = \frac{-\cos t + \frac{\pi}{2}}{t^2}$$

Example 3

$$y' = \frac{ty - t}{t^2 - 3t + 2}$$

$$y' = \frac{t(y-1)}{t^2 - 3t + 2}$$

$$\frac{1}{y-1}\,dy = \frac{t}{t^2 - 3t + 2}\,dt$$

$$\int \frac{1}{y-1}\,dy = \int \frac{t}{t^2 - 3t + 2}\,dt$$

$$\ln|y-1| = \int \frac{t}{t^2 - 3t + 2}\,dt$$

$$\int \frac{t}{t^2 - 3t + 2}\,dt = ?$$

$$\frac{t}{t^2 - 3t + 2} = \frac{t}{(t-2)(t-1)} = \frac{A}{t-2} + \frac{B}{t-1}$$

$$t = A(t-1) + B(t-2)$$

Let $t = 1$,

$$1 = B(1-2)$$

$$B = -1$$

Let $t = 2$,

$$2 = A(2-1)$$

$$A = 2$$

$$\frac{t}{t^2-3t+2} = \frac{2}{t-2} - \frac{1}{t-1}$$

$$\int \frac{t}{t^2-3t+2}\,dt = \int \frac{2}{t-2}\,dt - \int \frac{1}{t-1}\,dt$$

$$= 2\ln|t-2| - \ln|t-1| + C$$

$$\ln|y-1| = 2\ln|t-2| - \ln|t-1| + C$$

$$|y-1| = e^{2\ln|t-2|}\, e^{-\ln|t-1|}\, e^{C}$$

$$|y-1| = |t-2|^2\, |t-1|^{-1} \cdot e^{C}$$

$$y = 1 + C'\frac{(t-2)^2}{t-1}$$