Selected Solutions to Homework 5

Tim Smits

July 28, 2022

9.6.26 Find a fundamental solution set for $y' = Ay$ with $A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix}$.

Solution: One can check that the characteristic polynomial of A is $p_A(x) = x^3 + 5x^2 + 8x + 4$ which factors as $(x+1)(x+2)^2$. Therefore, the eigenvalues are $\lambda = -1, -2$ of multiplicity $1, 2$ respectively. An eigenvector for -1 is given by $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. The eigenspace for -2 is one dimensional, so we can only find a single linearly independent eigenvector, say $v_2 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$. This means we need a generalized eigenvector to find a third fundamental solution. To do this, we look at $(A+2I)^2 = \begin{pmatrix} -2 & 4 & -2 \\ -1 & 2 & -1 \\ 1 & -2 & 1 \end{pmatrix}$. If we can find a vector in the kernel of this matrix that's linearly independent from v_1 and v_2, we're good. Directly computing the kernel or otherwise, one may find that $v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is a generalized eigenvector and is linearly independent from v_1 and v_2. This yields the three fundamental solutions: $y_1(t) = e^{-t}v_1 = \begin{pmatrix} e^{-t} \\ e^{-t} \\ e^{-t} \end{pmatrix}$, $y_2(t) = e^{-2t}v_2 = \begin{pmatrix} -2e^{-2t} \\ -e^{-2t} \\ e^{-2t} \end{pmatrix}$, $y_3(t) = e^{-2t}(I + (A+2I)t)v_2 = \begin{pmatrix} -(t+1)e^{-2t} \\ -te^{-2t} \\ (1-t)e^{-2t} \end{pmatrix}$.

Alternatively, one may follow the "general algorithm" procedure outlined in the textbook and in lecture.

9.6.34 Two students computed two different fundamental solutions sets. Did one of the students make a mistake?

Solution: Neither student made a mistake when computing their fundamental solution sets. It's very important to recognize that the procedure for computing a fundamental solution set requires a *choice* of basis of generalized eigenvectors. If two students choose two different bases, barring computation errors afterwards, both fundamental solution sets are valid!

In this particular problem, you can actually recover the two bases that each student chose by examining their answers closely. Student one selected $\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ as their basis (these latter two can be seen because the fundamental solution coming from a generalized eigenvector of exponent 2 looks like $(I + (A - \lambda I)t)v$, so the term without the t in it is the vector), while student two selected $\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix}$ as their basis.

9.9.4 Solve $Y' = AY + F$ where $A = \begin{pmatrix} -3 & 10 \\ -3 & 8 \end{pmatrix}$ and $F = \begin{pmatrix} e^{-t} \\ e^{2t} \end{pmatrix}$.

Solution: We do this using variation of parameter. First, we need the fundamental matrix. To get that, we need to solve $Y' = AY$. One may check that the eigenvalues of A are $2, 3$ with corresponding eigenvectors $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ respectively. The fundamental solutions are therefore $y_1(t) = \begin{pmatrix} 2e^{2t} \\ e^{2t} \end{pmatrix}$ and $y_2(t) = \begin{pmatrix} 5e^{3t} \\ 3e^{3t} \end{pmatrix}$. This says $Y_f(t) = \begin{pmatrix} 2e^{2t} & 5e^{3t} \\ e^{2t} & 3e^{3t} \end{pmatrix}$. Now, variation of parameter says $v = \int Y_f(t)^{-1} F(t)\, dt$. We have $Y_f(t)^{-1} = \begin{pmatrix} 3e^{-2t} & -5e^{-2t} \\ -e^{-3t} & 2e^{3t} \end{pmatrix}$, and so $v = \int \begin{pmatrix} 3e^{-3t} - 5 \\ -4e^{-4t} + 2e^{5t} \end{pmatrix} dt = \begin{pmatrix} -e^{-3t} - 5t \\ e^{-4t} + \frac{2}{5}e^{5t} \end{pmatrix}$. A particular solution is then given by $Y_p(t) = Y_f(t)v(t)$, and the general solution is $c_1 y_1(t) + c_2 y_2(t) + Y_p(t)$.

2. The system

$$\frac{dR}{dt} = R(-R - S + 70)$$
$$\frac{dS}{dt} = S(-2R - S + a)$$

is a competing species model, where $\frac{dS}{dt}$ depends on some parameter $a \geq 0$.

(a) Find the two non-zero bifurcation values of a. *(Hint: these occur when the number of equilibrium points change.)*

(b) Describe the fates of the R and S populations before and after each of these bifurcations. Make sure your answers are justified mathematically by performing any necessary qualitative analyses of the model.

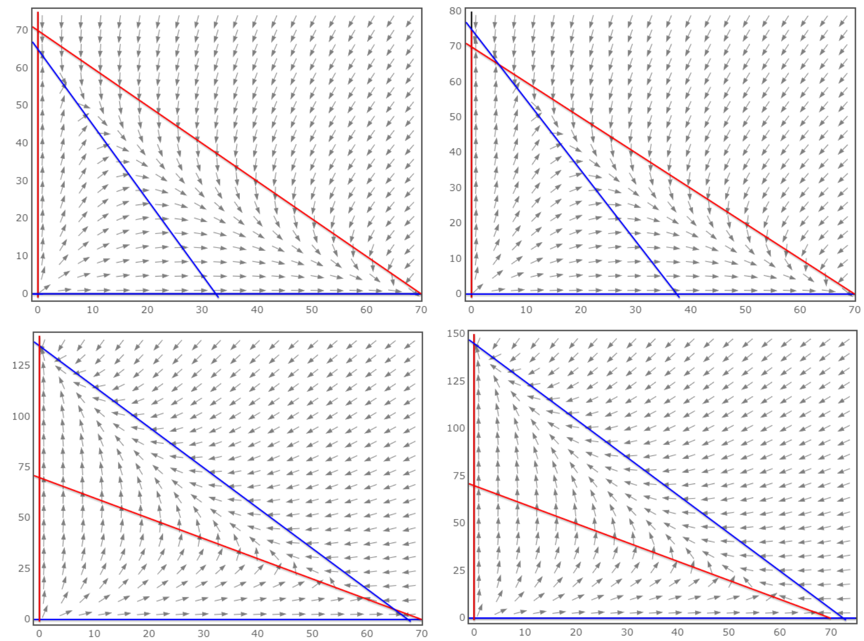
Solution:

(a) The equilibrium points of the model are $(0, 0)$, $(0, a)$, $(70, 0)$, and $(a - 70, 140 - a)$. So we see that we generally have four equilibrium points, except when $a = 0, a = 70$, and $a = 140$, where we only have three. These are the bifurcation values of a, so the non-zero ones are $a = 70, 140$.

(b) Pick values of a slightly before and after 70 and 140, say $a = 65, 75$ and $a = 135, 145$. We'll examine four different competing species models qualitative to decide what's going on with the populations. Now, you could also just pick a value of a in each of the three regions $0 < a < 70$, $70 < a < 140$, and $140 < a$. The overall outcomes remain the same if you perform the analysis this way.

However, I personally think it's a little easier to understand how the bifurcation works by doing four diagrams.

Attached below are phase portraits generated by a computer along with the corresponding nullclines. You can generate these pictures by hand by drawing nullclines and seeing which way solution trajectories have to go, in addition to classifying the local behavior at equilibrium points to help you gain information for making your sketches. However, for the sake of a solution, it's much easier to see what's happening like this.



These are the phase portraits for $a = 65, 75, 135, 145$ respectively. Let me describe what happens to the fates of the R, S populations respectively in each case:

- $a = 65$: All solutions with non-zero initial conditions get dragged towards the equilibrium point at $(70, 0)$. This corresponds to long term extinction of the S population.
- $a = 75$: There's now two possibilities here: solutions either get pulled towards $(70, 0)$ or $(0, 75)$ (the equilibrium at $(5, 65)$ is a saddle). The latter only happens for solutions that eventually enter the "small triangle" region in the upper left. This corresponds to long term extinction of either the R or S population, depending on where the initial condition starts.
- $a = 135$: Same scenario: all solutions tend to either $(70, 0)$ or $(0, 135)$ (once more, the equilibrium point at $(65, 5)$ is a saddle). The former behavior only happens for solutions that eventually enter the "small triangle" region in the lower right. In the long term, one population must always die out, depending on where the initial condition starts.
- $a = 145$: This is very similar to the first case now: we see that all solutions get pulled towards $(0, 140)$ so that the R population goes extinct in the long term.

To best visualize the bifurcation, the value of a corresponds to the S intercept of the blue line. So you can imagine increasing a as "pulling" out the blue line in the diagonal direction, causing the intercepts to slide up and right. As long as there is a triangular region between the two lines, it's possible for one species to outcompete the other depending on initial populations. The bifurcation values of $a = 70$ and $a = 140$ correspond to when the S and R intercepts of the red and blue lines coincide, respectively. They are the values when the overall outcome of the fate of the two species change. Passing through $a = 70$ causes the creation of the upper left triangular region, and passing through 140 causes the deletion of the lower right triangular region. I hope that this gives you a much deeper understanding of what is happening!