Selected Solutions to Homework 4

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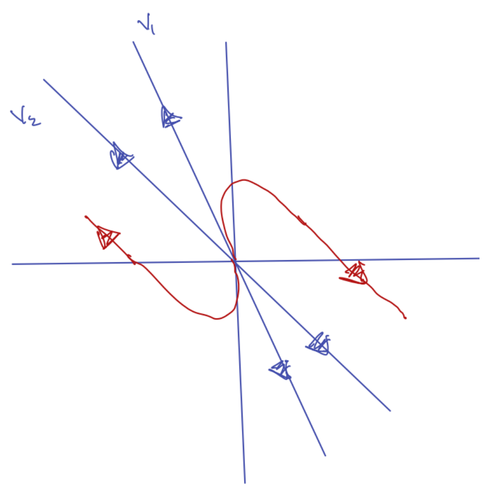
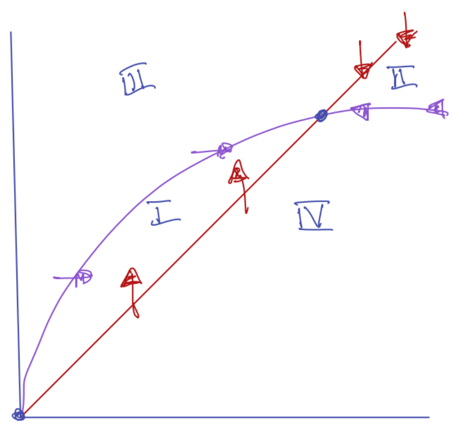
9.4.4 Classify the equilibrium points of the system $Y' = \begin{pmatrix} 8 & 3 \\ -6 & -1 \end{pmatrix} Y$, sketch the phase portrait by hand, and find the general solution.

Solution: The characteristic polynomial is $p_A(x) = x^2 - 7x + 10 = (x-5)(x-2)$. Thus, the eigenvalues are $\lambda = 2, 5$. An eigenvector for 2 is $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and an eigenvector for 5 is $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The general solution is then $Y(t) = c_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{5t}$. Since the two eigenvalues are positive, distinct real numbers, this means the phase portrait of the system is a nodal source. Since the trajectories must leave the origin and eventually become parallel to v_2 and they approach the origin tangent to v_1, we get the following picture. To determine which way it seems like solutions trajectories "rotate" in this case, pick a test point, say $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The vector field at this point is $Av = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$ which appears to be "clockwise".



10.3.14 Perform a global analysis in the first quadrant of the phase plane of the system $x' = y - x$, $y' = x - y^2$ to determine the trajectory of various solutions.

Solution: The x-nullcline is the line $y = x$ and the y-nullcline is the parabola $x = y^2$. The equilibrium points are where $y = x$ and $y = y^2$, i.e. $(0,0)$ and $(1,1)$. The nullclines in the first quadrant along with the corresponding directions of movement along the nullclines are sketched below. Now, I claim that all trajectories tend towards $(1, 1)$. In regions I and II this is obvious, because solutions can only move right/up and down/left respectively. In region IV, because $x > y$ we have $x' < 0$ and since $x > y^2$ we have $y' > 0$, so all movement must be left/up. This leaves three possibilities: either a trajectory enters region I or region II, in which case it gets forced towards $(1, 1)$, or the solution just approaches $(1, 1)$ while staying in IV. Region III similarly has $y > x$ so $x' > 0$ and $y^2 > x$ so $y' < 0$, so all movement is right/down. The same analysis shows that any trajectory starting in region III tends to $(1, 1)$.



1. Consider the one parameter family of linear systems

$$Y' = \begin{pmatrix} a & a^2 + a \\ 1 & a \end{pmatrix} Y$$

(a) Sketch the curve in the trace-determinant plane that comes from varying the parameter a.

(b) Determine all bifurcation values of a and describe the different types of phase portraits that are exhibited by this one parameter family.

Solution:

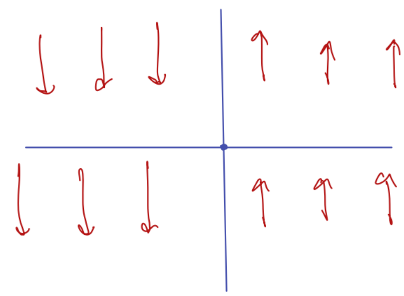
(a) The trace of the matrix is $T = 2a$ and the determinant is $D = -a$, so the curve cut out by this one parameter family is the line $D = -\frac{1}{2}T$.

(b) The bifurcation values happen when $D = -\frac{1}{2}T$ intersects the critical curve $T^2 = 4D$. This happens when $T^2 = -2T$, i.e. $T = 0, -2$. This corresponds to $a = 0, -1$ respectively, so these are the two bifurcation values of a. This gives rise to several cases:

- $a < -1$: this means $T < -2$ and $D > 1$, so we are underneath the left branch of the parabola, so the family is a nodal sink.
- $a = -1$: this happens when $T = -2$ and $D = 1$, which is on the left branch of the parabola. This is a degenerate nodal sink.
- $-1 < a < 0$: this means $-2 < T < 0$ and $0 < D < 1$, so we are above the left branch of the parabola, which says the family is a spiral sink.
- $a = 0$: this means $T = 0$ and $D = 0$. There's not a name for this case, but we can still sketch the phase portrait.

- $a > 0$: we have $T > 0$ and $D < 0$ so the family is a saddle.

The generic looking picture for each of these cases should be clear, with the exception of $a = 0$. (Make sure you understand how you would be able to sketch a *specific* phase portrait if you fixed a value of a!) To figure out what the phase portrait looks like for $a = 0$, note that this gives the system $Y' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} Y$, i.e. $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = x$. This says there is no x-direction movement, and all y-direction movement is proportional to the x-coordinate. Therefore, the phase portrait consists of vertical lines that move up in the right half plane, and down in the left half plane.



2. Consider the differential equation $y'' - y + y^3 = 0$.

(a) Convert the differential equation into a 2D system of the form $y' = f(y, v)$ and $v' = g(y, v)$ for some functions f, g. Find the equilibrium points, and classify their behavior.

(b) Find a function $H(y, v)$ such that $\partial H/\partial v = f(y, v)$ and $-\partial H/\partial y = g(y, v)$.

Solution:

(a) Set $y' = v$, so that $v' = y'' = y - y^3$. The equilibrium points of this system happen when $v = 0$ and $y - y^3 = 0$, i.e. $(0, 0)$, $(0, 1)$, $(0, -1)$. The Jacobian matrix of the system is $\begin{pmatrix} 0 & 1 \\ 1 - 3y^2 & 0 \end{pmatrix}$. At $(0, 0)$, this gives $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has trace 0 and determinant -1, which is a saddle. Since a saddle is generic, the non-linear system also has a saddle at $(0, 0)$. At $(0, 1)$ and $(0, -1)$ this is $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$, which has trace 0 and determinant 2, so it is a center. Since a center is non-generic, we can't necessarily conclude anything about the behavior of these equilibrium points for the non-linear system. However, by part (b) we see that the trajectory of solutions must follow the level curves of $H(y, v) = \frac{1}{2}v^2 - \frac{1}{2}y^2 + \frac{1}{4}y^4$. These level curves are *closed*, and the only equilibrium behavior where solutions are closed curves are centers. Therefore, this means $(0, 1)$ and $(0, -1)$ are actually centers for the non-linear system as well.

(b) We want $\partial H/\partial v = v$, so $H(y, v) = \frac{1}{2}v^2 + h(y)$ for some h. Taking a y-partial, this means $\partial H/\partial y = -(y - y^3) = h'(y)$, so $h(y) = -\frac{1}{2}y^2 + \frac{1}{4}y^4$ works. This says $H(y, v) = \frac{1}{2}v^2 - \frac{1}{2}y^2 + \frac{1}{4}y^4$ is a choice of H that works.

Remark: If you have a non-linear system $x' = f(x, y)$ and $y' = g(x, y)$ and you can find a function $H(x, y)$ such that $\partial H/\partial y = f(x, y)$ and $-\partial H/\partial x = g(x, y)$, then two things happen:

1. The trajectories of solutions to the system must follow along the level curves of $H(x, y)$.

2. The level curves of $H(x, y)$ are closed curves near equilibrium points where the linearization is a center.

This gives us a general technique for proving that a center remains a center for a non-linear system! If you have taken math 32A/B before, here's an explanation as to why this happens.

1. The vector field $X = (\partial H/\partial y, -\partial H/\partial x)$ tells us which way tangent vectors to solution trajectories point. This vector field is orthogonal to the gradient field ∇H, and the gradient is normal to level curves. Being orthogonal to the gradient therefore means X is *tangent* to level curves, so those give the trajectory.

2. If $f(x, y) = \partial H/\partial y$ and $g(x, y) = -\partial H/\partial x$, then at an equilibrium point (x_0, y_0) of the system the Jacobian matrix is $\begin{pmatrix} \partial^2 H/\partial y \partial x & \partial^2 H/\partial y^2 \\ -\partial^2 H/\partial x^2 & -\partial^2 H/\partial x \partial y \end{pmatrix}$, which has trace 0 because mixed partials are equal, and determinant $-(\partial H/\partial x \partial y)^2 + \partial^2 H/\partial x^2 \cdot \partial^2 H/\partial y^2$. The Hessian matrix of H is given by $\begin{pmatrix} \partial^2 H/\partial x^2 & \partial^2 H/\partial x \partial y \\ \partial^2 H/\partial y \partial x & \partial^2 H/\partial y^2 \end{pmatrix}$ which has the same determinant. If the linearization is a *center* at (x_0, y_0), then this determinant is positive. The second partial derivative test then says that H has a local max/min at (x_0, y_0), and so locally near this point the graph of H looks like a "bowl", pointing downwards or upwards respectively. Intersecting with a constant z-plane then gives level curves that look roughly like ellipses, which in particular are closed.