Selected Solutions to Homework 3

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4.5.38 Solve $y'' + 16y = e^{-4t} + 3\sin(4t)$.

Solution: First, we find a particular solution $y'' + 16y = e^{-4t}$. We guess $y_p(t) = Ae^{-4t}$. Plugging in, we get $16Ae^{-4t} + 16Ae^{-4t} = e^{-4t}$, so $A = \frac{1}{32}$. Next, we find a particular solution to $y'' + 16y = 3\sin(4t)$. We guess $y_p(t) = A\cos(4t) + B\sin(4t)$. Plugging in, $-16A\sin(4t) + 16B\cos(4t) = 3\sin(4t)$, so $A = -\frac{3}{16}$ and $B = 0$. This gives $y_p(t) = -\frac{3}{16}\cos(4t)$. The sum of these particular solutions then solves our equation, giving $y_p(t) = \frac{1}{32}e^{-4t} - \frac{3}{16}\cos(4t)$ as the solution.

9.2.20 Solve $Y' = AY$ with $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$.

Solution: The characteristic polynomial is $x^2 + 2x + 10$ which has roots $-1 \pm 3i$. Take $\lambda = -1+3i$ so $\bar{\lambda} = -1-3i$. To find an eigenvalue for λ, we solve $(A - \lambda I)v = 0$. This means solving $\begin{pmatrix} -1-(-1+3i) & 3 \\ -3 & -1-(-1+3i) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This says $-3ix + 3y = 0$ and $-3x - 3iy = 0$, so $y = ix$. We may then take $v = \begin{pmatrix} 1 \\ i \end{pmatrix}$ as an eigenvector, so $\bar{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector for $\bar{\lambda}$. The general solution is then $Y(t) = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+3i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-3i)t}$.

9.2.30 Solve $Y' = AY$ with $A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$.

Solution: The characteristic polynomial is $x^2 + 4x + 4 = (x+2)^2$, so $\lambda = -2$ is a repeated eigenvalue. To find an eigenvector, we solve $(A + 2I)v = 0$, i.e. $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This says $x = y$, so we can take $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ as an eigenvector. Now, we need to find a generalized eigenvector. To do this, we need to solve $(A + 2I)v_2 = v$, i.e. $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ this means $y - x = 1$, so we can choose $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The general solution is then $Y(t) = (c_1 + c_2 t)\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$.

3. Consider the IVP $y'' - (1 - y^2)y' + y = 0, \quad y(0) = 0, \quad y'(0) = 1$.

(a) Use the method outlined in class to convert the second order differential equation into a system of first order differential equations.

(b) Use Euler's method with step size $h = .1$ to approximate $y(1)$.

Solution:

(a) Set $v = y'$, so $v' = y'' = (1 - y^2)y' - y = (1 - y^2)v - y$. This gives the system
$$\begin{cases} y' &= v \\ v' &= (1 - y^2)v - y \end{cases}$$

(b) The iteration for Euler's method is given by $t_{k+1} = t_k + .1$, $y_{k+1} = y_k + v_k \cdot .1$, $v_{k+1} = v_k + ((1 - y_k^2)v_k - y_k) \cdot .1$. This yields the following table:

k	t_k	y_k	v_k
0	0	0	1
1	.1	.1	1.1
2	.2	.21	1.20
3	.3	.330	1.292
4	.4	.459	1.374
5	.5	.597	1.437
6	.6	.740	1.470
7	.7	.888	1.462
8	.8	1.034	1.405
9	.9	1.174	1.292
10	1.0	1.303	1.126

So that $y(1) \approx 1.303$.