Solutions to Homework 2

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2.7.30 Suppose that x is a solution to the initial value problem $x' = \frac{x^3-x}{1+t^2x^2}, \quad x(0) = 1/2$. Show that $0 < x(t) < 1$ for all t for which x is defined.

> **Solution:** Note that $f(t,x)$ is continuous everywhere, and $\frac{\partial f}{\partial t} = -\frac{2tx^3(x^2-1)}{(1+t^2x^2)^2}$ is also continuous everywhere. Therefore, any initial value problem of the form $x(a) = b$ for real numbers a, b has a unique solution. Now, note that $y_1(t) = 0$ and $y_2(t) = 1$ are solutions to $x' = f(t,x)$. Suppose that $x(t) \leq 0$ for some t. Since $x(0) = 1/2 > 0$, this would mean $x(t_0) = y_1(t_0)$ for some t_0 by the intermediate value theorem. However this is impossible, because this would violate the uniqueness of solution to the IVP $x' = f(t,x)$ with $x(t_0) = 0$. Therefore, $x(t) > 0$ for all t where x is defined. Similar reasoning shows $x(t) < 1$ for all t where x is defined.

2.9.10 Identify the equilibrium points and sketch the equilibrium solutions. Classify each equilibrium point as either stable or unstable.

> **Solution:** The equilibrium points are $-2, -1/2, 1, 2$ with stability of stable,unstable,stable,unstable respectively. The equilibrium solutions are just the constant functions corresponding to each point.

4.3.28 Find the solution to $y'' + 25y = 0, y(0) = 1, y'(0) = -1$.

> **Solution:** The characteristic polynomial is $\lambda^2 + 25\lambda$ which has roots $\pm 5i$. Therefore, the general solution is $y(t) = C_1 \cos(5t) + C_2 \sin(5t)$ for constants C_1, C_2. Since $y(0) = 1$, this means $C_1 = 1$. Taking a derivative, $y'(t) = -5\sin(5t) + 5C_2\cos(5t)$, so $C_2 = -1/5$. This gives $y(t) = \cos(5t) - 1/5\sin(5t)$.

1.

(a) Suppose that L fishing licenses are to be offered. Modify the logistic equation to produce a family of differential equations $\frac{dP}{dt} = f_L(P)$ depending on the parameter L that models the described harvesting scenario.

(b) Sketch a bifurcation diagram for your family of differential equations. Where, if anywhere, does bifurcation happen? Use your diagram to help you determine what the largest number of licenses L_M that can be issued without risk of eventually killing the entire fish population is. Explain why your choice is correct.

(c) Use a phase diagram to help you describe the long term behavior of the fish population, assuming that the L_M licenses would only be issued once the lake has stabilized at it's maximum fish population.

(d) As an ecologist, you realize that your model represents a "best case" environmental scenario. There are many real world factors that can also influence the population of the fish. For example, toxic waste from the local chemical plant might run into the lake if there is a particularly heavy rainstorm, killing off some fish. Given the possibility of unexpected changes in the fish population not accounted for in your model, what do you tell the department? Is it safe to recommend they issue L_M licenses, or should they go with less?

Solution:

(a) Each fisherman catches 3 fish per year and we issue L licenses, for a total harvest of $3L$ fish per year. This gives the new differential equation $\frac{dP}{dt} = .75(1 - \frac{P}{350})P - 3L$.

(b) The equilibrium points of $f_L(P)$ are given by $175 \pm \sqrt{8575 - 392L}$ by the quadratic formula. Bifurcation therefore happens when $L = \frac{8575}{392} = 21.875$. Note that $.75(1 - \frac{P}{350})P$ has maximum value of 65.625. Since we can't issue a fractional number of licenses, we need to decide if we round this up or down. If $L \geq 22$, then there are no equilibrium points (the term inside the square root is negative!) and $3L \geq 66$ so $\frac{dP}{dt} < 0$ always. This means the fish population is always decreasing, and must eventually die out. Therefore, we can issue out most 21 licenses, so $L_M = 21$.

(c) Taking $L_M = 21$, the model become $\frac{dP}{dt} = .75(1 - \frac{P}{350})P - 63$ which has equilibrium points 140 and 210. One can check that 210 is stable while 140 is unstable. The maximum long term fish population according to the logistic model is 350, so we have an initial condition $P(0) = 350$. This lies above 210, so in the long term, the fish population tends to 210 fish.

(d) Assuming we allow fishing with 21 licenses, long term the population stabilizes at 210 fish. As long as fishing happens while the fish population is above 140, the model says it will rebound back to 210 in the long term. Therefore, there's about 70 fish worth of "buffer room" for a disaster, or 33% of the total stable population size.

Whether or not you feel it's likely that an unforeseen factor will kill off at least 33% of the fish population at once while fishing is still allowed to happen in the lake is a personal judgment call – there's no "right answer" here. Personally, I think short of a very serious disaster (e.g. a large chemical spill, a hurricane, etc.) this is unlikely to happen, and in such a scenario I don't think fishing would be happening anyway. In the vacuum that is this problem, I'd probably say the 21 licenses are safe to hand out.

Really though, the point is to get you thinking and to interpret your model in the context of the scenario. If you're someone who will go into a job that requires modeling real world scenarios, there's not going to be a "right answer" when you have to make decisions! (Of course, in the real world, you would have plenty of contextual information to help you!).