Solutions to Homework 1

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2.4.20 Solve the IVP $y' = \cos(x) - y\sec(x)$ with $y(0) = 1$ and find the interval of existence.

Solution: The integrating factor is $\mu(x) = e^{\int \sec(x)\,dx} = e^{\ln|\sec(x)+\tan(x)|} = \ln|\sec(x)+\tan(x)|$. Moving the $y\sec(x)$ term over to the left and through and multiplying through says $\frac{d}{dx}[(\sec(x)+\tan(x))y] = (\sec(x)+\tan(x))\cos(x) = 1+\sin(x)$. Integrating and solving for y, we find $y(x) = \frac{x-\cos(x)+C}{\sec(x)+\tan(x)}$. Plugging in $y(0) = 1$ gives $C = 2$, so the solution is $y(x) = \frac{x-\cos(x)+2}{\sec(x)+\tan(x)}$. The interval of existence for the solution is the largest interval containing 0 where $y(x)$ is defined. Note that neither $\sec(x)$ nor $\tan(x)$ are defined at $-\pi/2$ or $\pi/2$, and there are no potential problems in between these values. Therefore, the interval of existence is $(-\pi/2, \pi/2)$.

2.6.20 Solve the differential equation $2xy^2 + 4x^3 + 2x^2y\frac{dy}{dx} = 0$.

Solution: We rewrite this as a differential form: $\omega = (2xy^2+4x^3)\,dx + 2x^2y\,dy = P\,dx + Q\,dy$. First, we check if ω is exact: $\frac{\partial P}{\partial y} = 4xy$ and $\frac{\partial Q}{\partial x} = 4xy$ so we're good. We want to find F such that $\omega = dF$. This means we must have $2xy^2 + 4x^3 = \frac{\partial F}{\partial x}$, so integrating gives $F(x,y) = x^2y^2+x^4+g(y)$. Differentiating gives $\frac{\partial F}{\partial y} = 2x^2y+g'(y)$. Since ω is exact, $\frac{\partial F}{\partial y} = 2x^2y$ so this means $g'(y) = 0$. We may then take $g(y) = 0$ as a solution to this equation, and so we can take $F(x,y) = x^2y^2 + x^4$. Therefore, the general solution is given by $x^2y^2 + x^4 = C$.

2.6.38 Solve the differential equation $\frac{dy}{dx} = \frac{y(x^2+y^2)}{xy^2-2x^3}$.

Solution: Note that this equation is homogeneous of degree 3. Set $y = vx$, so $dy = v\,dx + x\,dv$. Rewriting and substituting this in, $v\,dx + x\,dv = \frac{v^3+v}{v^2-2}\,dx$ so that $x\,dv = \frac{3v}{v^2-2}\,dx$. Separating, $(\frac{v}{3} - \frac{2}{3v})\,dv = \frac{1}{x}\,dx$ and so integrating yields $\frac{v^2}{6} - \frac{2}{3}\ln|v| = \ln|x| + C$. Plugging in $v = \frac{y}{x}$ and simplifying a bit yields $\frac{y^2}{6x^2} - \frac{2}{3}\ln|y| = \frac{1}{3}\ln|x| + C$ as our implicit solution.

2.

(a) Find an explicit formula for $A(t)$.

(b) Find an explicit formula for $T(t)$.

Solution:

(a) We have $A(2) = 64$ and $A(14) = 80$, so plugging into $A(t) = C\sin(\pi t/12) + D$ we get $64 = \frac{C}{2} + D$ and $80 = -\frac{C}{2} + D$. Solving this gives $C = -16$ and $D = 72$.

(b) From part (a), the differential equation we want to solve is $\frac{dT}{dt} = -.03(T + 16\sin(\pi t/12) - 72)$. The integrating factor is $\mu(t) = e^{\int .03\, dt} = e^{.03t}$. Moving the $-.03T$ term over to the left hand side and multiplying through, we get $\frac{d}{dt}(e^{.03t}T) = -.03e^{.03t}(16\sin(\pi t/12) - 72)$, so $e^{.03t}T = -\int .03e^{.03t} \cdot 16\sin(\pi t/12)\, dt + \int .03e^{.03t} \cdot 72\, dt$. The first integral can be computed via a standard "trick": see here. The other integral is easy. If you do the integration and solve for $T(t)$, you'll arrive at $T(t) = \frac{-144(9\sin(\pi t/12) + 25\pi\cos(\pi t/12))}{81 + 625\pi^2} + 72 + Ce^{-.03t}$. (If you wish, you can use one of the given conditions to solve for C, although it was fine if you just left the general solution.)