Math 33B
Homework 2
Due Friday, July 8, 2022

Do the following problems from each section of the textbook:

- 2.7: 4,8,10,30
- 2.9: 10,20,26
- 4.1: 20
- 4.3: 24,26,28

The following two sections are not part of this homework, but they **will** be on the list of topics for your midterm. If you would like to get a head start, these problems will be on homework 3:

- 4.5: 18,20,28,38,40
- 4.6: 4,10,14

Do the following additional problems:

1. The California Department of Fish and Wildlife has decided to allow fishing of a certain species of fish in a small lake, but they're not sure how many fishing licenses to allow. They've contacted you, local ecologist and hobbyist fisher, for help. You've determined based off their records that the population $P(t)$ of the fish species can be roughly modeled according to the logistic equation

$$\frac{dP}{dt} = .75(1 - \frac{P}{350})P.$$

Using your own fishing experience, you expect that the average fisher might catch about 3 of these fish per year (they're a hard catch!).

(a) Suppose that L fishing licenses are to be offered. Modify the logistic equation to produce a family of differential equations $\frac{dP}{dt} = f_L(P)$ depending on the parameter L that models the described harvesting scenario.

(b) Sketch a bifurcation diagram for your family of differential equations. Where, if anywhere, does bifurcation happen? Use your diagram to help you determine what the largest number of licenses L_M that can be issued without risk of eventually killing the entire fish population is. Explain why your choice is correct.

(c) Use a phase diagram to help you describe the long term behavior of the fish population, assuming that the L_M licenses would only be issued once the lake has stabilized at it's maximum fish population.

(d) As an ecologist, you realize that your model represents a "best case" environmental scenario. There are many real world factors that can also influence the population of the fish. For example, toxic waste from the local chemical plant might run into the lake if there is a particularly heavy rainstorm, killing off some fish. Given the possibility of unexpected changes in the fish population not accounted for in your model, what do you tell the department? Is it safe to recommend they issue L_M licenses, or should they go with less? *(Hint: think about how much "buffer room" your model has, and decide how bad of a disaster would be required to happen for the fish population to die off if fishing is allowed.)*

2. This problem will illustrate a technique for solving the homogeneous linear differential equation $y'' = p(t)y' + q(t)y$. Suppose that $y_1(t)$ and $y_2(t)$ are two solutions to such an equation.

   (a) Show that the Wronskian $W(t)$ of y_1 and y_2 satisfies the differential equation $W'(t) = p(t)W(t)$. Deduce that $W(t) = Ce^{P(t)}$ for some constant C, where $P(t)$ is a choice of anti-derivative of $p(t)$.

   (b) Show that y_1 satisfies the differential equation $y' - \frac{y_2'}{y_2}y = -\frac{W(t)}{y_2}$. This means that if one solution, say y_2, is known, you can solve a first order ODE to find another one.

   (c) Suppose I tell you that $y_2(t) = t^2$ is a solution to $y'' = -\frac{1}{t}y' + \frac{4}{t^2}y$. Use the method outlined in the previous parts to find a second solution $y_1(t)$. *(Hint: you can take the constant of integration when you solve the first order ODE to be 0, since you're only interested in one solution!)*

   (d) Explain why your solutions y_1 and y_2 are linearly independent, and write down the general solution to the second order ODE in part (c).