# Math 33B
# Differential Equations

## Midterm

**Directions:** Do the problems below. You have 24 hours to complete this exam, from 8:00 AM PST on Tuesday, July 12th to 8:00 AM PST on Wednesday, July 13th, by which time you must scan and upload your exam on Gradescope. You may use any course resources from our BruinLearn page, the textbook, or your notes, but you may not use other internet resources, nor may you discuss the exam with anyone other than me. Do not use methods that have not been covered in class. You may use a basic calculator. Show your work. Write full sentences when necessary.

Name: ______________________________

UID: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 10 | |
| 3 | 15 | |
| 4 | 13 | |
| Total: | 50 | |

1. (12 pts.) Find non-zero real numbers $a, b$ that make $\mu(x,y) = x^a e^{by}$ an integrating factor for the differential equation $(x^2+e^{-y})\,dx+(x^3+x^2y)\,dy = 0$, and use your integrating factor to find the general solution.

 **Solution:** Set $\omega = (x^2+e^{-y})\,dx + (x^3+x^2y)\,dy$. For $\mu(x,y)$ to be an integrating factor, we need $\mu\omega = dF$ for some function $F(x,y)$. We have $\mu\omega = (x^{a+2}e^{by} + x^a e^{(b-1)y})\,dx + (x^{a+3}e^{by} + x^{a+2}e^{by}y)\,dy$. Being exact means that $\frac{\partial}{\partial y}(x^{a+2}e^{by} + x^a e^{(b-1)y}) = \frac{\partial}{\partial x}(x^{a+3}e^{by} + x^{a+2}e^{by}y)$. Computing partials, this means that $bx^{a+2}e^{by} + (b-1)x^a e^{(b-1)y} = (a+3)x^{a+2}e^{by} + (a+2)x^{a+1}e^{(b+1)y}y$. In order for these to be equal, we require that $a = -2$ and $b = 1$, so $\mu(x,y) = e^y/x^2$ is an integrating factor. To solve the differential equation, we have $\mu\omega = (e^y + \frac{1}{x^2})\,dx + (xe^y + ye^y)\,dy$. Since this is exact, we have $\frac{\partial F}{\partial x} = e^y + \frac{1}{x^2}$ so $F(x,y) = xe^y - \frac{1}{x} + g(y)$ for some function $g(y)$. Taking a $y$-derivative, $\frac{\partial F}{\partial y} = xe^y + g'(y) = xe^y + ye^y$ by exactness. We may then take $g(y) = \int ye^y\,dy = (y-1)e^y$, so the general solution to the differential equation is given by $F(x,y) = xe^y - \frac{1}{x} + (y-1)e^y = C$ for constant $C$.

2. (10 pts.) Suppose that $y(t)$ is a solution to the initial value problem

$$\frac{dy}{dt} = \frac{y-t}{y^2+1} + 1, \quad y(0) = 1.$$

Is it possible that $y(1) = 0$? Justify your answer.

**Solution:** No, this is not possible. Note that $f(t,y) = \frac{y-t}{y^2+1} + 1$ is continuous everywhere, and $\frac{\partial f}{\partial y} = \frac{2ty-y^2+1}{(y^2+1)^2}$ is also continuous everywhere. Therefore, any IVP has a unique solution. Now, note that $y_2(t) = t$ is a solution to the differential equation. If $y(1) = 0$, then because $y_2(1) = 1$ we necessarily would need $y(t)$ and $y_2(t)$ to intersect somewhere in the interval $(0,1)$ (draw a picture!). However, this would violate the uniqueness theorem, so this cannot happen.

3. (15 pts.)

   (a) (10 pts.) Find the general solution to each differential equation.

   - $y'' - 2y' + y = te^t$
   - $t^2y'' - ty' + y = 0$, given that $y_2(t) = t$ is one solution.

   (b) (5 pts.) Does there exist a differential equation of the form $y'' + p(t)y' + q(t)y = 0$ with fundamental solutions $y_1(t) = t$ and $y_2(t) = t^2 + 1$ on the interval $(-2, 2)$? Justify your answer.

**Solution:**

(a)

- We use variation of parameter. The homogeneous equation is $y'' - 2y' + y = 0$ which has characteristic polynomial $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, which has a single repeated root. The homogeneous solution is therefore $y_h(t) = C_1e^t + C_2te^t$ for arbitrary $C_1, C_2$. Now, $W(t) = (e^t)(te^t)' - (e^t)'(te^t) = e^{2t}$. Plugging into the formula, we have $v_1(t) = -\int \frac{te^t}{e^{2t}} te^t\, dt = -\int t^2\, dt = -\frac{1}{3}t^3$ and $v_2(t) = \int \frac{e^t}{e^{2t}} te^t\, dt = \int t\, dt = \frac{1}{2}t^2$. This says a particular solution is $y_p(t) = v_1(t)e^t + v_2(t)te^t = \frac{1}{6}t^3e^t$. The general solution is therefore $y_h(t) + y_p(t) = C_1e^t + C_2te^t + \frac{1}{6}t^3e^t$.

- We follow the procedure from HW 2. The differential equation can be written as $y'' = \frac{1}{t}y' - \frac{1}{t^2}y$. We have $W(t) = Ce^{\int \frac{1}{t}\, dt} = Ct$. To find another solution $y_1(t)$, we solve $y' - \frac{1}{t}y = \frac{-Ct}{t}$, i.e. $y' - \frac{1}{t}y = -C$. This has integrating factor $\mu(t) = e^{-\int \frac{1}{t}\, dt} = \frac{1}{t}$, so multiplying yields $\frac{d}{dt}(\frac{1}{t}y) = \frac{-C}{t}$. Integrating and solving for $y(t)$, (taking the constant of integration to be 0 since we only need one solution), we find $y(t) = -Ct\ln(t)$. Take $y_1(t) = t\ln(t)$, then $y_1(t)$ and $y_2(t)$ are linearly independent because their ratio $\ln(t)$ is not constant. Therefore, the general solution is given by $y(t) = C_1t + C_2t\ln(t)$ for arbitrary $C_1, C_2$.

(b) No, this is not possible. The Wronskian of $y_1$ and $y_2$ is $(t)(t^2 - 1)' - (t)'(t^2 + 1) = t^2 - 1$, which vanishes at $-1, 1$ in the interval $(-2, 2)$. However, we have a theorem in the textbook which says the Wronskian of two solutions on an interval $I$ is either always 0 or never 0, so such a differential equation cannot exist.

**Remark:** Alternatively, one can explicitly determine the "candidate" differential equation as follows. Plugging $y_1(t) = t$ into $y'' + p(t)y' + q(t)y = 0$ yields $p(t) + tq(t) = 0$. Plugging $y_2(t) = t^2 + 1$ into this yields $2 + 2tp(t) + q(t)(t^2 + 1) = 0$, and so solving yields $p(t) = 2t/(1 - t^2)$, $q(t) = -2/(1 - t^2)$, yielding $y'' + \frac{2t}{1-t^2}y' - \frac{2}{1-t^2}y = 0$. From this you can also see that the interval of existence cannot contain $-1, 1$ because the coefficients are not defined there.

4. (13 pts.) Suppose that you know the population $P(t)$ ($t$ measured in years) of deer in a forest can be modeled according to the differential equation

$$\frac{dP}{dt} = P\left(1 - \frac{P}{125}\right)\left(\frac{P}{25} - 1\right)$$

   (a) (3 pts.) Draw the phase diagram for the differential equation and classify the stability of each equilibrium point. Sketch the graph of at least one solution between each pair of adjacent equilibrium solutions as well as above and below each equilibrium solution.

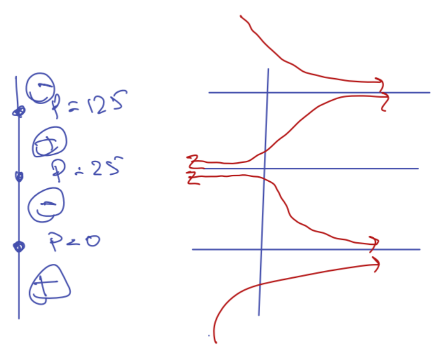
   Currently, there are around 175 deer in the forest and this is starting to cause trouble for the local farmers because the deer are coming into town and eating up all of their crops! To fix this problem, hunters have decided to start hunting some of the deer.

   (b) (5 pts.) Suppose that each year, $h\%$ of the deer population is allowed to be hunted. What is the maximal percentage of deer that can be safely hunted each year without risk of the population dying out? Justify your answer.

   (c) (5 pts.) What is the maximum percentage of deer that are allowed to be hunted each year if we do not wish the deer population to drop below 100? Justify your answer.

**Solution:**

(a) The equilibrium points are $0, 25, 125$. We see that 0 is stable, 25 is unstable, and 125 is stable.




(b) The family of differential equations $\frac{dP}{dt} = P(1 - \frac{P}{125})(\frac{P}{25} - 1) - hP$ models the harvesting scenario (here we write the percentage as a decimal). This can be written as $\frac{dP}{dt} = P((1 - \frac{P}{125})(\frac{P}{25} - 1) - h)$. There's always an equilibrium point of 0, and the equilibrium points from the other term are $75 \pm 25\sqrt{4 - 5h}$ by the quadratic formula. Therefore, bifurcation happens at $h = \frac{4}{5}$. If $h > \frac{4}{5}$, there is a single equilibrium point at 0, and $\frac{dP}{dt} < 0$ always because the maximum value of $(1 - \frac{P}{125})(\frac{P}{25} - 1)$ is $\frac{4}{5}$. If $h = \frac{4}{5}$, there is an equilibrium point at 75 which we see is semi-stable, so the 175 deer will tend to 75 long term. Therefore, we cannot hunt more than 80% of the deer population.

(c) We want to make sure that the long term deer population does not drop below 100. From the previous part, the non-zero equilibrium points of the model happen at $75 \pm 25\sqrt{4 - 5h}$. The larger one, $75 + 25\sqrt{4 - 5h}$ happens to stable, so the 175 deer population tends to this value long term. Therefore, we need $75 + 25\sqrt{4 - 5h} \geq 100$, so $\sqrt{4 - 5h} \geq 1$ yields $h \leq \frac{3}{5}$. Therefore, we may hunt at most 60% of the deer for the population to not drop below 100 deer.