Math 33B
Differential Equations

Final Exam

Directions: Do the problems below. You have 24 hours to complete this exam, from 8:00 AM PST on Thursday, July 26th to 8:00 AM PST on Friday, July 27th, by which time you must scan and upload your exam on Gradescope. You may use anything from our BruinLearn page, the textbook, or your notes. You may not use other internet resources (including the URLs that lead to outside the BruinLearn page), nor may you discuss the exam with anyone other than me. Do not use methods that have not been covered in class. You may use a basic calculator. Show your work or you will not get much credit. Write full sentences when necessary.

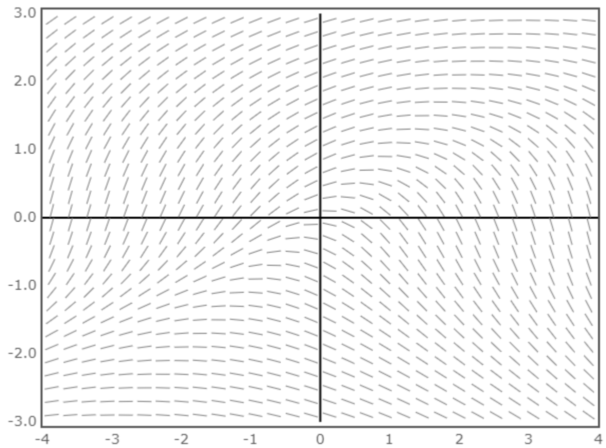
Name: ______________________

UID: ______________________

Question	Points	Score
1	12	
2	12	
3	12	
4	14	
5	10	
6	14	
7	16	
8	10	
Total:	100	

1. (12 pts.) True/False and Short answer. For True/False questions, give a quick justification for your answer.

   (a) (3 pts.) True or False: The slope field below comes from an autonomous differential equation.



   (b) (3 pts.) True or False: If y_0 is an equilibrium point of the non-linear 2×2 system $Y' = f(Y)$ and the Jacobian at y_0 has trace $2a^2$ and determinant a^4 for some non-zero real number a, then y_0 is an unstable equilibrium point.

   (c) (3 pts.) Find an integrating factor for $y\,dx + (2xy - e^{-2y})\,dy = 0$.

   (d) (3 pts.) Suppose you are told the characteristic polynomial of a homogeneous linear differential equation with constant coefficients is given by $p(x) = (x-1)^4(x^2+1)^3(x+5)$. Write down the general solution to the differential equation.

Solution:

(a) **False**. An autonomous differential equation has slope independent of time, so all slopes would look the same along horizontal lines.

(b) **True**. The characteristic polynomial of the Jacobian is $x^2 - 2a^2x + a^4 = (x-a^2)^2$. Both eigenvalues are positive, so by the stability theorem, y_0 is unstable.

(c) We have $\frac{1}{P}(P_y - Q_x) = \frac{1-2y}{y} = \frac{1}{y} - 2$. This only depends on y, so an integrating factor is found by solving $\mu' = -(\frac{1}{y} - 2)\mu$, which yields $\mu(y) = \frac{e^{2y}}{y}$.

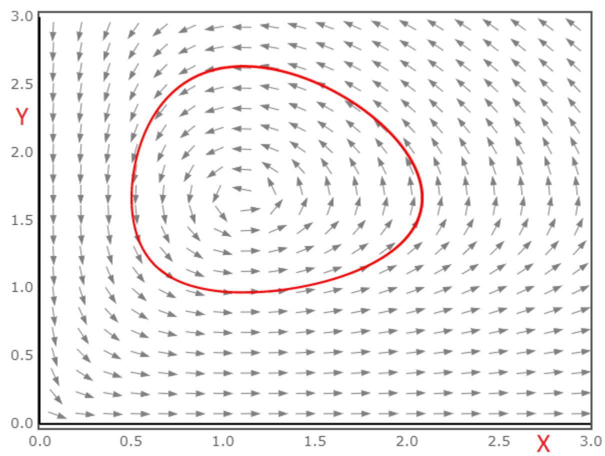
(d) The fundamental solutions coming from $\lambda = 1$ are $e^t, te^t, t^2e^t, t^3e^t$, the fudamental solutions coming from $\lambda = \pm i$ are $\cos(t), \sin(t), t\cos(t), t\sin(t), t^2\cos(t), t^2\sin(t)$, and the fundamental solution coming from $\lambda = -5$ is e^{-5t}. Taking a linear combination yields a general solution of $c_1e^t + c_2te^t + c_3t^2e^t + c_4t^3e^t + c_5\cos(t) + c_6\sin(t) + c_7t\cos(t) + c_8t\sin(t) + c_9t^2\cos(t) + c_{10}t^2\sin(t) + c_{11}e^{-5t}$.

2. (12 pts.) True/False and Short answer. For True/False questions, give a quick justification.

(a) (3 pts.) True or False: The IVP $y' = y^{4/5}, y(0) = 0$ has a unique solution.

(b) (3 pts.) True or False: Since the Wronskian of t^4 and $t^3|t|$ is 0 on $(-\infty, \infty)$, they must be linearly dependent functions on $(-\infty, \infty)$.

(c) (3 pts.) Suppose the phase portrait of the system $x' = f(x, y)$ and $y' = g(x, y)$ with initial condition $x(0) = 2, y(0) = 2$ is given below. Sketch graphs of $x(t)$ and $y(t)$. (You do not need to label the t-axis since you do not have that information, but get the shape of the graphs correct.)



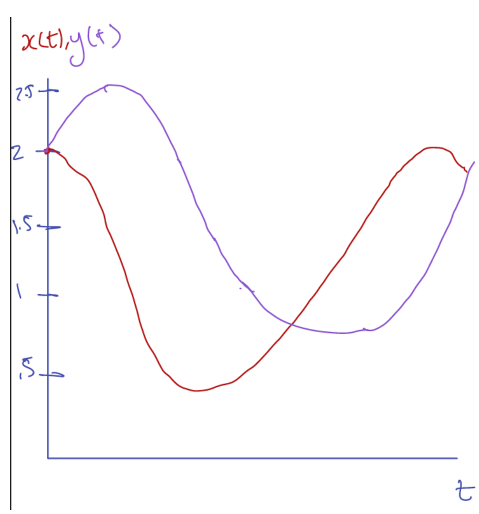
(d) (3 pts.) Find a first order autonomous differential equation with stable equilibrium points at 5, 10, unstable equilibrium point at 7, and semi-stable equilibrium point at -1.

Solution:

(a) **False**. Two solutions are $y_1(t) = 0$ and $y_2(t) = t^5/3125$.

(b) **False**. If $t^4 = ct^3|t|$ for some c, then plugging in 1 and -1 says $1 = c$ and $1 = -c$, which is not possible.

(c) The orbit in the phase plane is a closed curve, so the graphs of $x(t)$ and $y(t)$ will be periodic. Here are rough sketches of their behavior for a single period.



(d) One may take $y' = (y - 5)(y - 7)(10 - y)(y + 1)^2$.

3. (12 pts.) Find the general solution to each differential equation:

- $2xy\,dy + (\cos(x) + y^2)\,dx = 0$.
- $y'' - 4y' + 13y = e^{2t}\sec(3t)$.
- $t^2y'' - t(t+2)y' + (t+2)y = 0$, given that $y(t) = t$ is a solution.

Solution:

- We have $P_y - Q_x = 2y - 2y = 0$, so this equation is exact. Integrating P with respect to x, we have $F(x,y) = \sin(x) + xy^2 + g(y)$ for some function g. Taking a y-derivative, we have $2xy + g'(y) = Q = 2y$, so $g'(y) = 0$ says $g(y) = 0$ works. This gives a general solution of $\sin(x) + xy^2 = C$ for constant C.
- The homogeneous solution is given by solving $y'' - 4y' + 13y = 0$ which has characteristic polynomial $x^2 - 4x + 13$. The roots are $2 \pm 3i$, for a general solution of $y_h(t) = c_1e^{2t}\cos(3t) + c_2e^{2t}\sin(3t)$. For a particular solution, use variation of parameter. The Wronskian $W(t)$ of $y_1(t) = e^{2t}\cos(3t)$ and $y_2(t) = e^{2t}\sin(3t)$ is $3e^{4t}$, so $v_1(t) = \int \frac{-e^{2t}\sin(3t)}{3e^{4t}}e^{2t}\sec(3t)\,dt = \frac{1}{9}\ln|\cos(3t)|$ and $v_2(t) = \int \frac{e^{2t}\cos(3t)}{3e^{4t}}e^{2t}\sec(3t)\,dt = \frac{t}{3}$. This gives $y_p(t) = y_1(t)v_1(t) + y_2(t)v_2(t)$. The general solution is then $y(t) = y_h(t) + y_p(t) = c_1e^{2t}\cos(3t) + c_2e^{2t}\sin(3t) + \frac{1}{9}e^{2t}\cos(3t)\ln|\cos(3t)| + \frac{1}{3}te^{2t}\sin(3t)$.
- Write the equation as $y'' = (1 + \frac{2}{t})y' + (\frac{1}{t} + \frac{2}{t^2})y$. The Wronskian is given by $W(t) = Ce^{\int 1 + \frac{2}{t}\,dt} = Ct^2e^t$. To find a second solution, we solve $y' - \frac{1}{t}y = -Cte^t$. This has integrating factor $\mu(t) = \frac{1}{t}$, so $\frac{d}{dt}[\frac{1}{t}y] = -Ce^t$. Integrating and solving for $y(t)$ (taking the constant of integration to be 0) says $y(t) = -Cte^t$ solves this, so we can take $y_2(t) = te^t$ as another solution. This is clearly linearly independent from $y_1(t) = t$, so the general solution is $y(t) = c_1t + c_2te^t$.

4. (14 pts.) Consider the one parameter family of linear systems

$$Y' = \begin{pmatrix} a & \frac{1}{4}(a^2 - 18) \\ 1 & 0 \end{pmatrix} Y.$$

(a) (5 pts.) Find all bifurcation values of a and describe the different types of phase portraits that are exhibited by this family as a varies.

(b) (9 pts.) Sketch a phase portrait and write down the general solution for the system for $a = -3, 2, 4$. Make sure you plot enough trajectories to make it clear what the trajectory through any arbitrary point might look like!

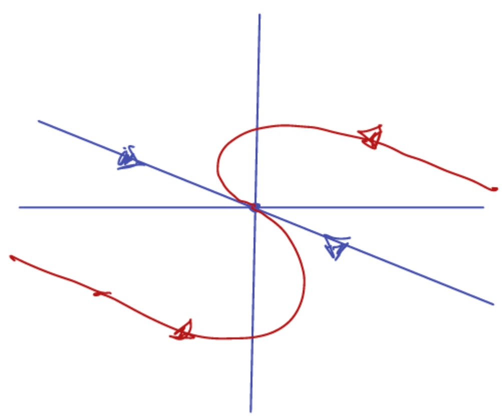
Solution:

(a) The trace of A is $T = a$ and the determinant is $D = \frac{1}{4}(18 - a^2)$, so in the trace-determinant plane, the family cuts out the curve $D = \frac{1}{4}(18-T^2)$. There are five bifurcation values: the two intersections with the T-axis, the intersection with the D-axis, and the two intersections with the critical curve $T^2 = 4D$. This gives $a = \pm 3\sqrt{2}, 0, \pm 3$. The curve $D = \frac{1}{4}(18 - T^2)$ crosses through all regions of the trace-determinant plane that can exhibit different behaviors, so *all* equilibrium types appear from this family! More specifically:

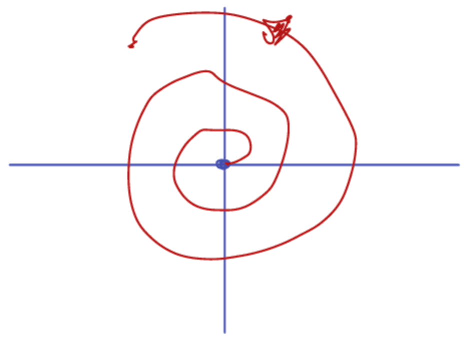
- $a < -3\sqrt{2}$ or $a > 3\sqrt{2}$: these are saddles.
- $a = -3\sqrt{2}$ or $a = 3\sqrt{2}$: these are a stable saddle-node and an unstable saddle-node respectively.
- $-3\sqrt{2} < a < -3$ or $3 < a < 3\sqrt{2}$: these are nodal sinks and nodal sources, respectively.
- $a = -3$ or $a = 3$: These are a degenerate nodal sink/source respectively.
- $a = 0$: this is a center.

(b) I will spare you the computational details.

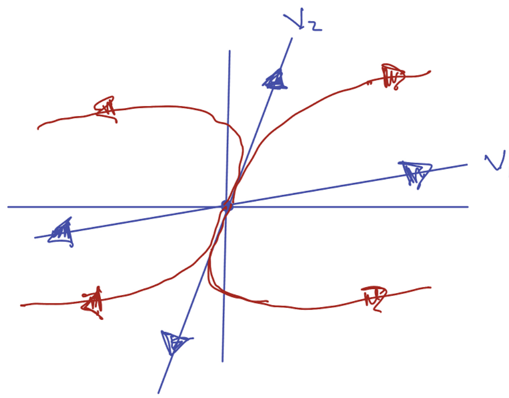
- $a = -3$: The matrix is $\begin{pmatrix} -3 & -\frac{9}{4} \\ 1 & 0 \end{pmatrix}$. There is a single eigenvalue of $\lambda = -\frac{3}{2}$ with eigenvector $v = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. A generalized eigenvector is $v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, so the general solution is $y(t) = c_1 e^{-3/2t} v + c_2 e^{-3/2t}(tv + v_2)$. At $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the vector field is $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$, which indicates that solutions follow a "counterclockwise" looking path as indicated below.



- $a = 2$: The matrix is $\begin{pmatrix} 2 & -\frac{7}{2} \\ 1 & 0 \end{pmatrix}$. The eigenvalues are $1 \pm \frac{\sqrt{10}}{2} i$. An eigenvector is $v = \begin{pmatrix} 2 + i\sqrt{10} \\ 2 \end{pmatrix}$. The general solution is $y(t) = c_1 e^{(1+i\sqrt{10}/2)t} v + c_2 e^{(1-i\sqrt{10}/2)t} \bar{v}$. At $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the vector field is $\begin{pmatrix} -1/2 \\ 0 \end{pmatrix}$, which indicates that solutions follow a "counterclockwise" looking path as indicated below.



- $a = 4$: The matrix is $\begin{pmatrix} 4 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$. The eigenvalues are $2 \pm \frac{1}{2}\sqrt{14}$. Eigenvectors are $v_1 = \begin{pmatrix} 4 + \sqrt{14} \\ 2 \end{pmatrix}$ and $v_2 = v_1 = \begin{pmatrix} 4 - \sqrt{14} \\ 2 \end{pmatrix}$, and the general solution is $y(t) = c_1 e^{(2+\frac{\sqrt{14}}{2})t} v_1 + c_2 e^{(2-\frac{\sqrt{14}}{2})t} v_2$.



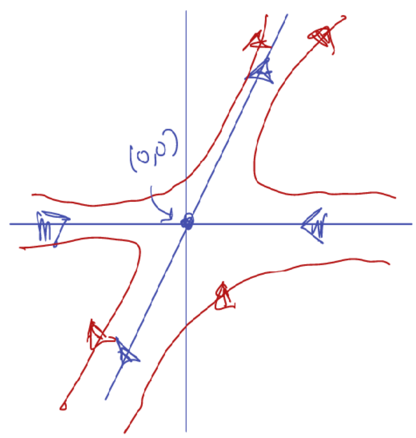
5. (10 pts.) Consider the non-linear system of differential equations

$$\frac{dx}{dt} = -x + y + x^2$$
$$\frac{dy}{dt} = y - 2xy$$

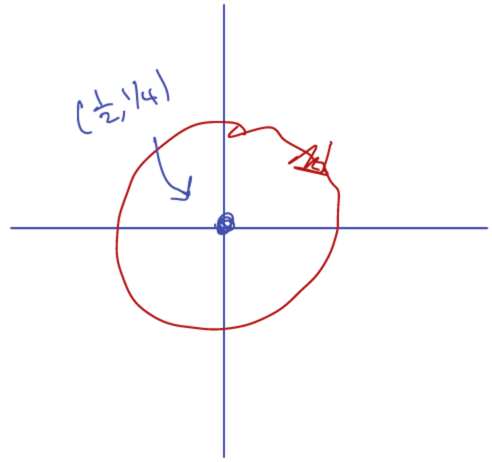
Draw a phase portrait of the non-linear system locally near each equilibrium point. You must prove carefully that your phase portraits have the correct behavior type to receive full credit.

Solution: The equilibrium points are given by $(0,0)$, $(1/2, 1/4)$, and $(1,0)$. The Jacobian matrix of the system is $J(x,y) = \begin{pmatrix} -1+2x & 1 \\ -2y & 1-2x \end{pmatrix}$. We analyze the Jacobian at each of these:

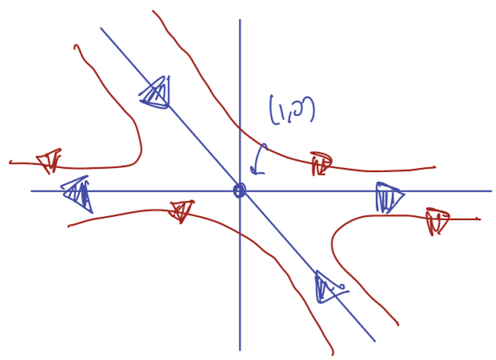
- $(0,0)$: This is $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ which has trace 0 and determinant -1. This is a saddle, which is generic, so the nonlinear system is a saddle near $(0,0)$. Eigenvalues and eigenvectors of the Jacobian matrix are $\lambda = -1, 1$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ respectively.



- $(1/2, 1/4)$: this is $\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}$ which has trace 0 and determinant $\frac{1}{2}$. This is a center, which is not generic, so we can't use the usual theorem. We look for a function $H(x,y)$ such that $\frac{\partial H}{\partial y} = -x+y+x^2$ and $-\frac{\partial H}{\partial x} = y - 2xy$. Integrating the first condition says $H(x,y) = -xy + \frac{1}{2}y^2 + x^2 y + g(x)$ for some g. Taking an x-derivative, this gives $-\frac{\partial H}{\partial x} = y - 2xy - g'(x) = y - 2xy$, so $g'(x) = 0$ says $g(x) = 0$ works. This says $H(x,y) = -xy + \frac{1}{2}y^2 + x^2 y$ works, so by the HW 4 remark this proves that the nonlinear system is a center near $(1/2, 1/4)$. At $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the vector field is $\begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$, which indicates that solutions follow a "clockwise" looking path as indicated below.



- $(1,0)$: This is $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ which once again is a saddle, so the nonlinear system has a saddle near $(1,0)$. The eigenvalues and eigenvectors of the Jacobian matrix are $\lambda = -1, 1$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ respectively.



6. (14 pts.) Consider two animal species R and S that inhabit a forest. Their populations $R(t)$ and $S(t)$ (t measured in years) can be modeled by the system of differential equations

$$\frac{dR}{dt} = 7R - R^2 - RS$$
$$\frac{dS}{dt} = -5S + RS$$

(a) (2 pts.) Does this model represent a predator-prey scenario or competing species scenario? Justify your answer.

(b) (2 pts.) Hunters from the local town have been entering the forest and hunting the S population at a rate of S^2 per year. Modify the equation for $\frac{dS}{dt}$ to account for this new scenario, and write down the new system of differential equations.

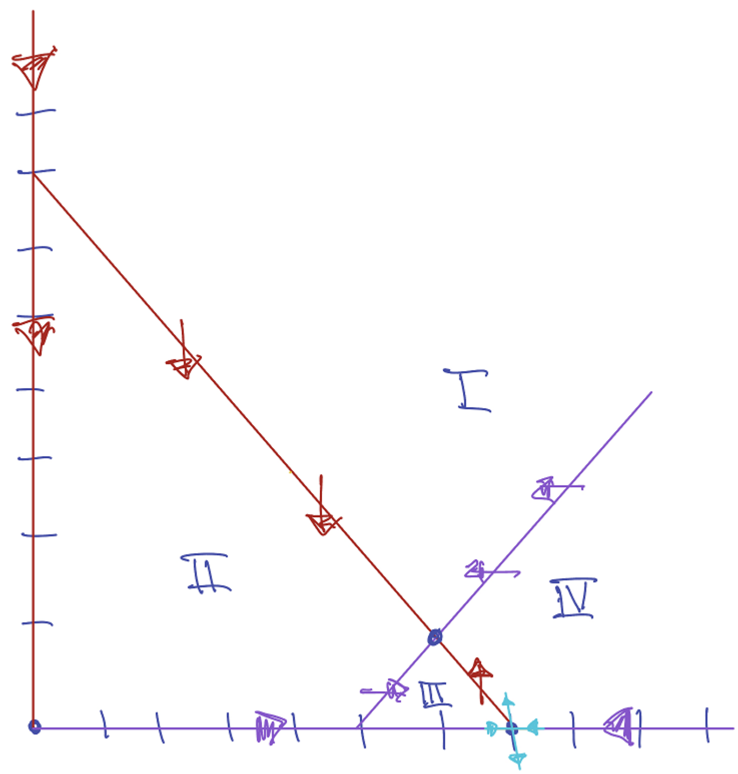
(c) (10 pts.)With the new model, determine the long term fates of the R and S populations. For which initial conditions do both species survive? One goes extinct? Both go extinct? Make sure to justify your answer, backing it up with local/global qualitative analysis of the model.

Solution:

(a) This is a predator-prey scenario. In the absence of the R population, the S population grows like $S' = -5S$, which exponentially decays, so they die out. In the absence of the S population, the R population grows like $R' = 7R(1 - \frac{R}{7})$ which is logistic growth, so they will survive.

(b) The new equation for S' is $S' = -S + RS - S^2$, so the new system is $R' = 7R - R^2 - RS$ and $S' = -5S + RS - S^2$.

(c) The equilibrium points of the model in the first quadrant are $(0, 0)$, $(6, 1)$, and $(7, 0)$. The Jacobian matrix is $J(x, y) = \begin{pmatrix} 7 - 2R - S & -R \\ S & -5 + R - 2S \end{pmatrix}$. At $(0, 0)$ and $(7, 0)$ these are saddles, and at $(6, 1)$ this is a nodal sink. Since all these behaviors are generic, the nonlinear system has the same behavior near these points. The R-nullclines are the lines $R = 0$ and $S = 7 - R$, while the S-nullclines are the lines $S = 0$ and $S = R - 5$. The eigenvalues and eigenvectors of $J(7, 0)$ are $\lambda = -7, 2$ and $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -7 \\ 9 \end{pmatrix}$. Performing a global analysis and adding in this local information yields the following picture:



Given any initial condition with $R, S > 0$ the solution curve can move through the indicated regions in increasing order. The presence of a saddle at $(7, 0)$ along with the direction of the eigenvectors should hopefully make it clear that if trajectory enters region III and crosses into IV, it cannot stray too far from the equilibrium point and therefore must get "pulled in" toward $(6, 1)$ because it's a sink. Other equilibrium points cannot be approached because they are saddles, so in particular extinction is never a concern.

7. (16 pts.) Consider the linear system $Y' = AY$ with $A = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) (12 pts.) Compute the fundamental matrix $Y_f(t)$ for this linear system.

(b) (4 pts.) Here is a useful fact: $e^{At} = Y_f(t)Y_f(0)^{-1}$. Use this fact to compute e^{At} and solve the IVP $Y' = AY$ with $Y(0) = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$.

(c) (Bonus) (2 pts.) Prove the above fact using the uniqueness theorem for linear systems.

Solution:

(a) The characteristic polynomial is $(x-3)^2(x-5)$, which gives eigenvalues of $\lambda = 3, 5$ of multiplicities $2, 1$ respectively. An eigenvector for $\lambda = 5$ is given by $v_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Now compute $(A-3I)^2 = \begin{pmatrix} 0 & 8 & 16 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{pmatrix}$. One choice of basis for the kernel is $v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$. These yield fundamental solutions of $Y_1(t) = e^{5t}v_1$, $Y_2(t) = e^3t(I + (A - 3I)t)v_2$ and $Y_3(t) = e^{3t}(I + (A - 3I)t)v_2$. Explicitly writing these out and making them columns, the fundamental matrix is $Y_f(t) = \begin{pmatrix} 2e^{5t} & e^{3t} & -3te^{3t} \\ e^{5t} & 0 & -2e^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix}$.

(b) We have $Y_f(0) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. Computing $Y_f(0)^{-1}$ by whatever your favorite method is yields $Y_f(0)^{-1} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -2 & -4 \\ 0 & 0 & 1 \end{pmatrix}$, so $e^{At} = \begin{pmatrix} e^{3t} & 2e^{5t} - 2e^{3t} & 4e^{5t} - 4e^{3t} - 3te^{3t} \\ 0 & e^{5t} & 2e^{5t} - 2e^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix}$. The solution to the IVP is given by $e^{At}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$.

(c) Set $X(t) = Y_f(t)Y_f(0)^{-1}$. Note that $X'(t) = Y_f'(t)Y_f(0)^{-1} = AY_f(t)Y_f(0)^{-1} = AX(t)$. Also, note that $(e^{At})' = Ae^{At}$. Since $X(0) = I$, this means for any vector v, the IVP $Y' = AY, Y(0) = v$ has solutions $X(t)v$ and $e^{At}v$. By the uniqueness theorem for linear systems, this means $X(t)v = e^{At}v$ for all vectors v. This then means $X(t) = e^{At}$ as desired.

8. (10 pts.) Consider the IVP $y''' = t + y^2, \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -1$. Use Euler's method with step size $h = .5$ to estimate $y(2)$, rounded to 3 decimal places. Your answer should include a table containing all of the appropriate values you used to compute your estimate.

Solution: If we convert the ODE into a system, we get the following:

$$\begin{aligned} y' &= v \\ v' &= w \\ w' &= t + y^2 \end{aligned}$$

which we can perform Euler's method on. We have $y_0 = 1$, $v_0 = 1$, and $w_0 = -1$, with $t_0 = 0$ and $h = .5$. This yields the iterates $y_{k+1} = y_k + \frac{1}{2} \cdot v_k$, $v_{k+1} = v_k + \frac{1}{2} \cdot w_k$, and $w_{k+1} = w_k + \frac{1}{2} \cdot (t_k + y_k^2)$, and $t_{k+1} = t_k + \frac{1}{2}$. This yields the following table:

k	t_k	y_k	v_k	w_k
0	0	1	1	-1
1	.5	1.5	.5	$-.5$
2	1	1.75	.25	.875
3	1.5	1.875	.6875	2.90625
4	2	2.21875	2.140625	5.4140625

So that $y(2) \approx 2.219$.