Math 33B
Differential Equations

Final Exam

Directions: Do the problems below. You have 24 hours to complete this exam, from 8:00 AM PST on Thursday, July 26th to 8:00 AM PST on Friday, July 27th, by which time you must scan and upload your exam on Gradescope. You may use anything from our BruinLearn page, the textbook, or your notes. You may not use other internet resources (including the URLs that lead to outside the BruinLearn page), nor may you discuss the exam with anyone other than me. Do not use methods that have not been covered in class. You may use a basic calculator. Show your work or you will not get much credit. Write full sentences when necessary.

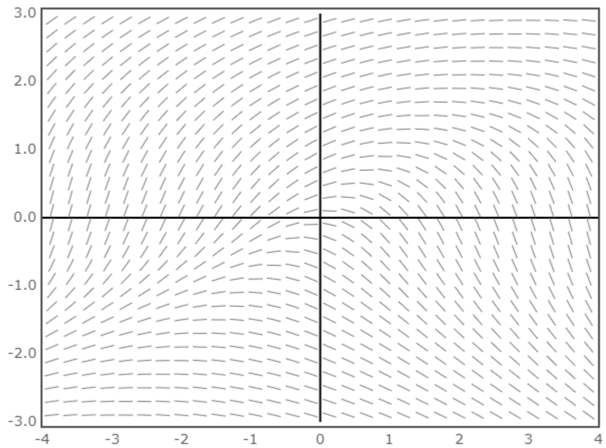
Name: ____________________________

UID: ______________________________

Question	Points	Score
1	12	
2	12	
3	12	
4	14	
5	10	
6	14	
7	16	
8	10	
Total:	100	

1. (12 pts.) True/False and Short answer. For True/False questions, give a quick justification for your answer.

   (a) (3 pts.) True or False: The slope field below comes from an autonomous differential equation.



   (b) (3 pts.) True or False: If y_0 is an equilibrium point of the non-linear 2×2 system $Y' = f(Y)$ and the Jacobian at y_0 has trace $2a^2$ and determinant a^4 for some non-zero real number a, then y_0 is an unstable equilibrium point.

   (c) (3 pts.) Find an integrating factor for $y\,dx + (2xy - e^{-2y})\,dy = 0$.

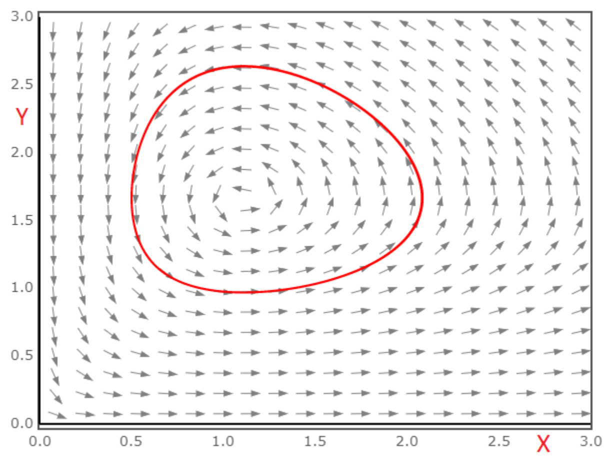
   (d) (3 pts.) Suppose you are told the characteristic polynomial of a homogeneous linear differential equation with constant coefficients is given by $p(x) = (x-1)^4(x^2+1)^3(x+5)$. Write down the general solution to the differential equation.

2. (12 pts.) True/False and Short answer. For True/False questions, give a quick justification.

   (a) (3 pts.) True or False: The IVP $y' = y^{4/5}, y(0) = 0$ has a unique solution.

   (b) (3 pts.) True or False: Since the Wronskian of t^4 and $t^3|t|$ is 0 on $(-\infty, \infty)$, they must be linearly dependent functions on $(-\infty, \infty)$.

   (c) (3 pts.) Suppose the phase portrait of the system $x' = f(x, y)$ and $y' = g(x, y)$ with initial condition $x(0) = 2, y(0) = 2$ is given below. Sketch graphs of $x(t)$ and $y(t)$. (You do not need to label the t-axis since you do not have that information, but get the shape of the graphs correct.)



   (d) (3 pts.) Find a first order autonomous differential equation with stable equilibrium points at $5, 10$, unstable equilibrium point at 7, and semi-stable equilibrium point at -1.

3. (12 pts.) Find the general solution to each differential equation:

- $2xy\,dy + (\cos(x) + y^2)\,dx = 0$.
- $y'' - 4y' + 13y = e^{2t}\sec(3t)$.
- $t^2y'' - t(t+2)y' + (t+2)y = 0$, given that $y(t) = t$ is a solution.

4. (14 pts.) Consider the one parameter family of linear systems

$$Y' = \begin{pmatrix} a & \frac{1}{4}(a^2 - 18) \\ 1 & 0 \end{pmatrix} Y.$$

(a) (5 pts.) Find all bifurcation values of a and describe the different types of phase portraits that are exhibited by this family as a varies.

(b) (9 pts.) Sketch a phase portrait and write down the general solution for the system for $a = -3, 2, 4$. Make sure you plot enough trajectories to make it clear what the trajectory through any arbitrary point might look like!

5. (10 pts.) Consider the non-linear system of differential equations

$$\frac{dx}{dt} = -x + y + x^2$$
$$\frac{dy}{dt} = y - 2xy$$

Draw a phase portrait of the non-linear system locally near each equilibrium point. You must prove carefully that your phase portraits have the correct behavior type to receive full credit.

6. (14 pts.) Consider two animal species R and S that inhabit a forest. Their populations $R(t)$ and $S(t)$ (t measured in years) can be modeled by the system of differential equations

$$\frac{dR}{dt} = 7R - R^2 - RS$$
$$\frac{dS}{dt} = -5S + RS$$

   (a) (2 pts.) Does this model represent a predator-prey scenario or competing species scenario? Justify your answer.

   (b) (2 pts.) Hunters from the local town have been entering the forest and hunting the S population at a rate of S^2 per year. Modify the equation for $\frac{dS}{dt}$ to account for this new scenario, and write down the new system of differential equations.

   (c) (10 pts.)With the new model, determine the long term fates of the R and S populations. For which initial conditions do both species survive? One goes extinct? Both go extinct? Make sure to justify your answer, backing it up with local/global qualitative analysis of the model.

7. (16 pts.) Consider the linear system $Y' = AY$ with $A = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 3 \end{pmatrix}$.

   (a) (12 pts.) Compute the fundamental matrix $Y_f(t)$ for this linear system.

   (b) (4 pts.) Here is a useful fact: $e^{At} = Y_f(t)Y_f(0)^{-1}$. Use this fact to compute e^{At} and solve the IVP $Y' = AY$ with $Y(0) = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$.

   (c) (Bonus) (2 pts.) Prove the above fact using the uniqueness theorem for linear systems.

8. (10 pts.) Consider the IVP $y''' = t + y^2, \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -1$. Use Euler's method with step size $h = .5$ to estimate $y(2)$, rounded to 3 decimal places. Your answer should include a table containing all of the appropriate values you used to compute your estimate.