Midterm 2 Practice
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1. For the following statements, indicate if they are true or false.

   (a) If $A = QR$ is the QR-factorization of A, then $AA^t = RR^t$

   (b) Let Proj_S be an orthogonal projection onto some subspace S of $\mathbb{R}^n$. If P is the matrix of this projection, then P is orthogonal.

   (c) If $T : \mathbb{R}^8 \to \mathbb{R}^5$ is a linear transformation such that $\dim(\ker(T)^\perp) = 3$, then T is surjective.

   (d) If A is an $n \times n$ matrix with $\|Au\| = 1$ for all unit vectors $u \in \mathbb{R}^n$, then A is orthogonal.

2. Give an example of the following concepts.

   (a) A 4×4 matrix A with $\text{Im}(A) = \ker(A)$.

   (b) A 4×4 matrix A with non-zero entries such that $\text{Im}(A)^\perp = \ker(A)$.

   (c) A 3×3 matrix A with non-zero entries such that A is orthogonal and $A = A^t$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $T(x, y) = (x + 2y, 4x + 3y)$, and let A be the matrix of T with respect to the standard basis.

   (a) Is there a basis β of $\mathbb{R}^2$ such $[T]_\beta = \begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}$? If so, find one. If not, explain why.

   (b) Is there a basis β of $\mathbb{R}^2$ such that $[T]_\beta = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$? If so, find one. If not, explain why.

4. Let v and w be vectors in $\mathbb{R}^2$.

   (a) Prove that $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$. What is a geometric interpretation of this equation? Does your proof still work in $\mathbb{R}^n$? (if not, find one that does!)

   (b) Show that $v \cdot w = \frac{1}{4}(\|v + w\|^2 - \|v - w\|^2)$. Again, your proof should hold in $\mathbb{R}^n$. (This says angles can be defined in terms of length!)

   (c) Suppose that $\|v - w\| = 3$, $\|v\| = 2$, and $\|w\| = 5$. What is $\|v + w\|$?

   (d) Suppose that we didn't know $\|v - w\| = 3$. What are the minimum and maximum possible values for $\|v + w\|$? When does equality hold in each case?

   (e) Give examples of vectors v, w with non-zero entries that satisfy the conditions of c).

5. Let A, B be $n \times n$ matrices with $B^t A = 0$.

   (a) Show that $\text{Im}(A) \perp \text{Im}(B)$.

   (b) Let P be an $n \times n$ matrix with $P^2 = P$. Prove that $\text{Im}(I - P) = \ker(P)$.

   (c) Now suppose that $P^2 = P$ and $P = P^t$. Show that $\text{Im}(P) \perp \ker(P)$.

6. Let $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

   (a) Compute the QR factorization of A.

(b) Let M be an $n \times m$ matrix with $\text{rank}(M) = m$. Suppose that $M = QR$ is the QR-decomposition of M, and x^* is the least squares solution to $Mx = b$. Write x^* in terms of Q and R. (This relates the seemingly disjoint sections 5.2 and 5.4)

(c) With $b = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, use part b) to compute the least squares solution x^* to $Ax = b$.

7. Let $A = \begin{pmatrix} 1 & -1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \\ 1 & -1 & \sqrt{2} \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

   (a) Find orthonormal bases for $\text{Im}(A)$ and $\ker(A)$.

   (b) Minimize $\|Ax - b\|$ such that $x \in \mathbb{R}^3$, and find a vector x such that $\|Ax - b\|$ achieves this minimal value.

   (c) Compute P, the matrix of $\text{Proj}_S(x)$ where $S = \ker(A)$.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation with $[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 5 \end{pmatrix}$, where $\beta = \{(1, 1, 1), (0, 1, 1), (0, 0, 1)\}$. Let A be the matrix of T with respect to the standard basis.

   (a) Compute the change of basis matrix $S_{\mathcal{E} \to \beta}$ and $[v]_\beta$ for $v = (5, -2, 3)$. Use this to write v as a linear combination of the basis vectors of β.

   (b) Compute A using the change of basis formula.

9. (a) Let A be an $n \times m$ matrix. Prove that $\dim(\ker(A)) - \dim(\ker(A^t)) = m - n$.

   (b) Let A be an $n \times n$ orthogonal matrix and B an $n \times m$ matrix. Show that $\text{rank}(B^t) + \dim(\ker(AB)) = m$.

10. Let A be an $n \times n$ matrix and $T(x) = Ax$ the corresponding linear transformation. Suppose that $A^n = 0$ but $A^{n-1} \neq 0$. Let $v \in \mathbb{R}^n$ such that $Av \neq 0$.

   (a) Prove that $\beta = \{A^{n-1}v, A^{n-2}v, \ldots, Av, v\}$ is a basis of $\mathbb{R}^n$.

   (b) Compute $[T]_\beta$, the matrix of T with respect to β.

   (c) Find a 3×3 matrix A with $A^3 = 0$ but $A^2 \neq 0$.

Extra: Suppose that P is an $n \times n$ matrix with $P^2 = P$ and $\text{Im}(P)^\perp = \ker(P)$. Prove that $P = P^t$. (This is the converse to $5(c)$. This says a projection matrix is symmetric if and only if it's an *orthogonal* projection).

Solutions

1. (a) False; If $A = QR$ then $A^t = R^tQ^t$ and $AA^t = QRR^tQ^t \neq RR^t$, unless $Q = I$. This is only possible if A is upper triangular, but obviously not all matrices are of this form.

   (b) False; An orthogonal matrix Q satisfies $Q^t = Q^{-1}$, so in particular, is invertible. An orthogonal projection is not invertible!

   (c) False; Since $\dim(\ker(T)^\perp) = 3$ and $\ker(T)^\perp = \mathrm{Im}(T^t)$, since $\mathrm{rank}(A^t) = \mathrm{rank}(A)$, this says $\dim(\mathrm{Im}(T^t)) = \dim(\mathrm{Im}(T)) = 3$. Since $3 \neq 5$, this says T is not surjective.

   (d) True; any vector x can be written as $x = \|x\|\frac{x}{\|x\|}$. Then $Ax = \|x\|A(\frac{x}{\|x\|})$, so $\|Ax\| = \|x\|\|A(\frac{x}{\|x\|})\| = \|x\|$ because $\frac{x}{\|x\|}$ is a unit vector. This says A is orthogonal.

2. (a) By rank-nullity, if A is such a matrix, then $\mathrm{rank}(A) = 2$ and $\dim(\ker(A)) = 2$. Since $\mathrm{Im}(A)$ is spanned by the columns of A, A must have two linearly independent columns. The matrix A is determined entirely by what it does on the vectors e_1, e_2, e_3, e_4, because Ae_i returns the i-th column of A. The easiest way foward is the following: pick two columns of A to be 0, say $Ae_3 = 0$ and $Ae_4 = 0$. Since $\ker(A) = \mathrm{Im}(A)$, this says $\mathrm{Im}(A) = \ker(A) = \mathrm{Span}\{e_3, e_4\}$ so we could take $Ae_1 = e_3$ and $Ae_2 = e_4$, and this would work. This matrix A is given by $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

   (b) Geometrically, a matrix with $\mathrm{Im}(A)^\perp = \ker(A)$ is an orthogonal projection. Pick your favorite line L in the direction of a vector with non-zero entries, then the matrix of the orthogonal projection onto L will have the desired property (or project onto some other subspace if you so desire). For example, with $L = \mathrm{Span}\{u\}$ for $u = (1/2, 1/2, 1/2, 1/2)$, we have $P = uu^t = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$.

   (c) Since A is orthogonal, $A^{-1} = A^t$ so this say $A^{-1} = A$, i.e. $A^2 = I$. Recall that such matrices come from reflections, so we can construct A by reflecting around any line L in the direction of a vector with non-zero entries (or reflecting around a plane if you so choose). If A is the matrix of the reflection, recall we have the relation $A = 2P - I$, where P is the projection. For example, take $L = \mathrm{Span}\{u\}$ where $u = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. Then $A = 2P - I = 2uu^t - I = \begin{pmatrix} \frac{-1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{-1}{3} \end{pmatrix}$ has the desired property.

3. (a) We have $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. If such a basis β existed, let $S = S_{\beta \to \mathcal{E}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the change of basis matrix. The change of basis formula would say $[T]_\beta = S^{-1}AS$, i.e. that $S[T]_\beta = AS$. This says $\begin{pmatrix} 3a - 3b & -a + b \\ 3c - 3d & -c + d \end{pmatrix} = \begin{pmatrix} a + 2c & b + 2d \\ 4a + 3c & 4b + 3d \end{pmatrix}$. Equating gives the system $\begin{cases} 3a - 3b = a + 2c \\ -a + b = b + 2d \\ 3c - 3d = 4a + 3c \\ -c + d = 4b + 3d \end{cases}$

   The second equation says $a = -2d$ and the third equation says $a = -\frac{3}{4}d$, so $a = d = 0$. The first and last equation then say $c = -\frac{3}{2}b$ and $c = -4b$, so $b = c = 0$. However, $S = 0$ is not a valid change of basis matrix, so this is not possible.

   Alternatively, we can easily see the answer is no because changing basis does not change the rank of T. However, $\mathrm{rank}(A) = 2$ while $\mathrm{rank}([T]_\beta) = 1$, so no such basis exists.

(b) Doing the same approach gives the system $\begin{cases} 5a = a + 2c \\ -b = b + 2d \\ 5c = 4a + 3c \\ -d = 4b + 3d \end{cases}$

Solving gives $2a = c$ and $b = -d$, so we may pick, for example, $a = 1$, $c = 2$ and $b = -1$,$d = 1$. So $S_{\beta\to\mathcal{E}} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$. Since the columns of S are just the vectors in this basis β, we have $\beta = \{(1, 2), (-1, 1)\}$ is the desired basis.

4. (a) We have $\|v + w\|^2 = (v + w) \cdot (v + w) = v \cdot v + 2(v \cdot w) + w \cdot w = \|v\|^2 + 2(v \cdot w) + \|w\|^2$. Similarly, $\|v - w\|^2 = v \cdot v - 2(v \cdot w) + w \cdot w = \|v\|^2 - 2(v \cdot w) + \|w\|^2$ so adding gives $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

   We can interpret this geometrically by letting v and w be the sides of a parallelogram, so the diagonals have length $\|v + w\|$ and $\|v - w\|$. This is then a statement relating the length squared of the diagonals to the length squared of the sides.

   (b) From $\|v + w\|^2 = (v + w) \cdot (v + w) = v \cdot v + 2(v \cdot w) + w \cdot w = \|v\|^2 + 2(v \cdot w) + \|w\|^2$ and $\|v - w\|^2 = v \cdot v - 2(v \cdot w) + w \cdot w = \|v\|^2 - 2(v \cdot w) + \|w\|^2$, we subtract instead of add to find $4(v \cdot w) = \|v + w\|^2 - \|v - w\|^2$.

   (c) Plug in to part a) to find $\|v + w\| = 7$.

   (d) We have $\|v+w\|^2 = \|v\|^2+2(v\cdot w)+\|w\|^2$, and $v\cdot w = \|v\|\|w\| \cos(\theta)$ where θ is the angle between v and w. Since $-1 \le \cos(\theta) \le 1$, we find $\|v + w\|^2 \le \|v\|^2 + 2\|v\|\|w\| + \|w\|^2 = (\|v\|+\|w\|)^2$, so $\|v+w\| \le \|v\|+\|w\|$. Similarly, we find $\|v+w\| \ge \Big|\|v\|-\|w\|\Big|$. Using our numbers, we find $3 \le \|v + w\| \le 7$. Equality holds whenever $\cos(\theta) = \pm 1$: $\cos(\theta) = +1$ happens when v and w are parallel and in the same direction, and $\cos(\theta) = -1$ happens when v and w are parallel but in opposite directions.

   (e) Pick any two vectors v and w with v parallel to w going in the same direction that have the right lengths. For example, $v = (2/\sqrt{2}, 2/\sqrt{2})$ and $w = (5/\sqrt{2}, 5/\sqrt{2})$.

5. (a) If $x \in \mathrm{Im}(A)$ and $y \in \mathrm{Im}(B)$, write $x = Av$ and $y = Bw$ for some vectors v, w. Then $x \cdot y = (Av) \cdot (Bw) = w^tB^tAv = 0$ because $B^tA = 0$. This says $\mathrm{Im}(A) \perp \mathrm{Im}(B)$.

   (b) If $x \in \mathrm{Im}(I - P)$, then write $x = (I - P)v$ for some v. Then $Px = P(I - P)v = Pv - P^2v = Pv - Pv = 0$. This says $x \in \ker(P)$, so $\mathrm{Im}(I - P) \subset \ker(P)$. If $x \in \ker(P)$, then $Px = 0$, so $(I - P)x = Ix = x$ says $x \in \mathrm{Im}(I - P)$, so $\ker(P) \subset \mathrm{Im}(I - P)$ gives $\mathrm{Im}(I - P) = \ker(P)$.

   (c) We have $(I - P)^tP = (I - P^t)P = (I - P)P = P - P^2 = 0$. By part a), this says $\mathrm{Im}(I - P) \perp \mathrm{Im}(P)$. By part b), we have $\mathrm{Im}(I - P) = \ker(P)$, so $\mathrm{Im}(P) \perp \ker(P)$.

6. (a) Doing Gram-Schmidt on the columns, we have $u_1 = v_1 = (0, 0, 1, 0)$, $v_2^\perp = v_2 - (v_2 \cdot u_1)u_1 = v_2 - u_1 = (0, 1, 0, 0)$ so $u_2 = (0, 1, 0, 0)$ and $v_3^\perp = v_3 - (v_3 \cdot u_1)u)1 - (v_3 \cdot u_2)u_2 = v_3 - u_1 - u_2 = (1, 0, 0, 0)$, so $u_3 = (1, 0, 0, 0)$. This says $Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We also have $v_1 = u_1$, $v_2 = u_1 + u_2$ and $v_3 = u_1 + u_2 + u_3$ so $R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

   (b) The least squares solution to $Mx = b$ is given by solving $M^tMx^* = M^tb$. Since $M = QR$, plugging in says we need to solve $(R^tQ^t)(QR)x^* = R^tQ^tb$. Since $Q^tQ = I$, this says $R^tRx^* = R^tQ^tb$. Now, $\mathrm{rank}(M) = \mathrm{rank}(M^tM) = \mathrm{rank}(R^tR) = \mathrm{rank}(R) = m$, so in particular, this says R is invertible. Thus, $x^* = (R^tR)^{-1}R^tQ^tb = R^{-1}Q^tb$.

(c) We have $R^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ and $Q^t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ so $x^* = R^{-1}Q^t b = (1, 2, -1)$.

7. (a) It's clear that $\mathrm{Im}(A) = \mathrm{Span}\{(1,1,1)\}$ so an orthonormal basis is given by $\{(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})\}$. Vectors in the kernel must lie in the plane $x - y + z\sqrt{2} = 0$, so a basis of this is given by $\{(1,1,0), (2, 0, -\sqrt{2})\}$. Running Gram-Schmidt on this set gives us the orthonormal basis $\{(1/\sqrt{2}, 1/\sqrt{2}, 0), (1/2, -1/2, -\sqrt{2}/2)\}$.

   (b) The minimal value is given by $\|b^\perp\|$ where $b^\perp = b - \mathrm{Proj}_{\mathrm{Im}(A)}(b)$. Since $\mathrm{Im}(A) = \mathrm{Span}\{u\}$ for $u = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, we have $\mathrm{Proj}_{\mathrm{Im}(A)}(b) = (b \cdot u)u = (1/3, 1/3, 1/3)$, so $b^\perp = (2/3, -1/3, -1/3)$ gives $\|b^\perp\| = 2/3$. To find a minimizer, we find a least squares solution. We need to solve $A^tAx^* = A^tb$, which is equivalent to solving $\begin{pmatrix} 3 & -3 & 2\sqrt{2} \\ -3 & 3 & -3\sqrt{2} \\ 3\sqrt{2} & -3\sqrt{2} & 6 \end{pmatrix} x^* = \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix}$. Using row reduction to solve the system, we see $x^* = (4/3, 1, 0)$ is a choice works.

   (c) The matrix P is given by $P = u_1u_1^t + u_2u_2^t$ where u_1 and u_2 are the vectors in the orthonormal basis of $\ker(A)$ from part a). We then find $P = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{\sqrt{2}}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{1}{2} \end{pmatrix}$.

8. (a) We have $S_{\mathcal{E}\to\beta} = (S_{\beta\to\mathcal{E}})^{-1}$ where $S_{\beta\to\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Inverting says $S_{\beta\to\mathcal{E}} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. Recall that $[v]_\beta = S_{\mathcal{E}\to\beta}v$, so we find $[v]_\beta = (5, -7, 5)$. This then says $v = 5(1,1,1) - 7(0,1,1) + 5(0,0,1)$ by reading off the coordinates.

   (b) The change of basis formula says
   $$A = S_{\beta\to\mathcal{E}}[T]_\beta S_{\mathcal{E}\to\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

9. (a) By rank-nullity, $\mathrm{rank}(A) + \dim(\ker(A)) = m$ and $\mathrm{rank}(A^t) + \dim(\ker(A^t)) = n$. Since $\mathrm{rank}(A) = \mathrm{rank}(A^t)$, subtracting says $\dim(\ker(A)) - \dim(\ker(A^t)) = m - n$.

   (b) Since $\ker(M) = \ker(M^tM)$ holds for any matrix M, setting $M = AB$ says $\ker(AB) = \ker((AB)^t(AB)) = \ker(B^tA^tAB) = \ker(B^tB) = \ker(B)$ because A is orthogonal. Since $\mathrm{rank}(B^t) = \mathrm{rank}(B)$, we see $\mathrm{rank}(B^t) + \dim(\ker(AB)) = \mathrm{rank}(B) + \dim(\ker(B)) = m$ by rank-nullity.

   Alternatively, A is invertible since A is orthogonal, so we have $\ker(AB) = \ker(B)$ because $ABx = 0 \iff Bx = 0$ because we can multiply by A or A^{-1} accordingly. We then proceed the same way.

10. (a) We show that β is linearly independent, and then it's a basis. Suppose that $c_{n-1}A^{n-1}v + \ldots + c_1Av + c_0v = 0$. Multiply through by A^{n-1}, so that all terms die off except the last, and we are left with $c_0A^{n-1}v = 0$. Since $A^{n-1}v \neq 0$ by assumption, this says $c_0 = 0$. We then see that multiplying by A^{n-2} will show that $c_1 = 0$. Repeating this process will show all $c_i = 0$, so that β is a linearly independent set.

    (b) Recall that $[T]_\beta = \begin{pmatrix} | & | & & | \\ [T(v_1)]_\beta & [T(v_2)]_\beta & \ldots & [T(v_n)]_\beta \\ | & | & & | \end{pmatrix}$ where $\beta = \{v_1, \ldots, v_n\}$. Here we take $v_i = A^{n-i}v$, so $T(v_i) = A(A^{n-i}v) = A^{n-(i-1)}v$. This says $T(v_1) = 0$ and $T(v_i) = v_{i-1}$ for $2 \leq i \leq n$, so that $[T(v_1)]_\beta = 0$ if $i = 1$ and $[T(v_i)]_\beta = e_{i-1}$ for $2 \leq i \leq n$. This then says $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$.

(c) The previous part says $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ works.

Extra: Since $\mathrm{Im}(P) \perp \ker(P)$ and $\ker(P) = \mathrm{Im}(I-P)$, this says $\mathrm{Im}(P) \perp \mathrm{Im}(I-P)$. So $(Px)\cdot((I-P)y) = 0$ for all $x, y \in \mathbb{R}^n$. This says $((I-P)y)^t Px = y^t(I-P^t)Px = ((I-P^t)Px)\cdot y = 0$ for all x, y. In particular, choosing $y = (I-P^t)Px$ says $\|(I-P^t)Px\|^2 = 0$ for all $x \in \mathbb{R}^n$, so that $(I-P^t)P = 0$. This says $P = P^tP$, and taking a transpose says $P^t = P^tP$ so that $P = P^t$ as desired.