Midterm 1 Practice

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Vectors are written as rows to save space. Interpret them as columns.

1. For the following statements, indicate if they are true or false.
   (a) The set $W = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 0\}$ is a subspace of $\mathbb{R}^3$.
   (b) For an $n \times n$ matrix A, if $A^2 = 0$, then $I_n + A$ is invertible.
   (c) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a surjective linear transformation with $n \leq m$. Then T is invertible.
   (d) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

2. Give an example of the following concepts.
   (a) A 3×3 system of equations that corresponds geometrically to two overlapping planes with a third one parallel.
   (b) A 4×3 matrix A such that the map $T(\vec{x}) = A\vec{x}$ is injective.
   (c) A basis of $\mathbb{R}^3$ with all basis vectors having entries 1 or -1.
   (d) A 3×3 matrix A with $A^2 \neq I_3$ and $A^4 = I_3$.

3. Consider the following system of equations: $\begin{cases} x + 3y + 2z + t = a \\ 2x + 5y + 3z = a \\ x - z = -b \\ y + z = b \end{cases}$
   (a) For what values of a, b does a solution exist? How many such solutions are there?
   (b) For the above values of a, b, solve the system of equations. What is a geometric description of the solution set?

4. Consider the following system of equations: $\begin{cases} x + 2y + 3z = 7 \\ 2x + 3y + 4z = 6 \\ 4x + 4y + 5z = 8 \end{cases}$
   (a) Translate the system of equations into a matrix equation of the form $A\vec{x} = \vec{b}$.
   (b) Explain why the matrix A is invertible, and compute A^{-1}. What does this tell you about the solution set to the system?
   (c) Solve the system of equations. What is a geometric description of the solution set?

5. Find all matrices B such that $AB = BA$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined as follows. First, orthogonally project a vector $\vec{x}$ onto the line $y = \frac{1}{\sqrt{3}}x$. Then, rotate counter-clockwise by an angle of $\pi/3$. Next, reflect across the x-axis, and finally, scale by a factor of 3.
   (a) Let A be the matrix of T. Let P be the matrix of the orthogonal projection, R be the matrix of the rotation, S be the matrix of the reflection, and D be the matrix of the scaling. Write down A in terms of D, P, R and S.

(b) Without doing any computation, describe $\mathrm{Im}(T)$ and $\ker(T)$. Is T invertible? (Hint: draw a picture!)

(c) Explicitly compute the matrix A, and use it to verify your answers in part (b).

7. Let $S = \{(1, -1, 1), (1, 4, 5), (6, 8, 10), (40, 33, 48)\}$.

   (a) Without doing any computation, explain why S is a linearly dependent set of vectors.

   (b) Write $(40, 33, 48)$ as a linear combination of the other 3 vectors.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(x, y, z) = (x + y - z, 2x + 2y - 2z, 2x + 3y - 5z)$. Find bases for $\mathrm{Im}(T)$ and $\ker(T)$, clearly stating their dimensions. Give a geometric description for each subspace.

9. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $T(3, 1) = (2, 3)$ and $T(1, 2) = (9, 1)$. Find the matrix A of T.

10. Let $V = \mathrm{span}\{(1, -1, 3, 1), (2, 3, 4, -5)\}$.

   (a) Find a 4×4 matrix A with $\mathrm{Im}(A) = V$.

   (b) What is the relationship between the rows of A and vectors in $\ker(A)$?

   (c) For your matrix A, write down its transpose A^t. What does the above say is the relationship between vectors in $\ker(A)$ and vectors in $\mathrm{Im}(A^t)$?

   (d) Find $\ker(A^t)$. What is the relationship between $\ker(A^t)$ and the subspace V?

Extra: Let A be a 2×2 matrix.

(a) Show that $\ker(A) \subseteq \ker(A^2)$ and $\mathrm{Im}(A^2) \subseteq \mathrm{Im}(A)$.

(b) Suppose $A^3 = 0$. Show that $A^2 = 0$.

Solutions

1. (a) False; W is not closed under addition. $(3,4,5) \in W$ and $(5,12,13) \in W$ but $(8,16,18) \notin W$.

   (b) True; $(I_n - A)(I_n + A) = (I_n + A)(I_n - A) = I_n - A^2 = I_n$ says $(I_n + A)$ is invertible with inverse $I_n - A$.

   (c) True; since T is surjective, this says $n \geq m$, and from $n \leq m$ this says $n = m$. A surjective map from $\mathbb{R}^n \to \mathbb{R}^n$ is necessarily invertible by the invertible matrix theorem.

   (d) False; $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$!

2. (a) The system of equations $x + y + z = 1$, $2x + 2y + 2z = 2$, $x + y + z = 0$ is one such example.

   (b) The map $T(\vec{x})$ is injective if A has full rank, so pick any 4×3 matrix A with rank 3. An example is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

   (c) Take $\beta = \{(1,-1,1), (1,1,-1), (-1,1,1)\}$. One can check using row reduction that the matrix $A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ has full rank, so the columns are linearly independent. Since $\dim(\mathbb{R}^3) = 3$, this says they actually form a basis.

   (d) Consider a counter-clockwise rotation R by $\pi/2$ around the z-axis, which geometrically will satisfy the desired property, so if A is the matrix of R, then this matrix will work. To compute the matrix of the rotation, compute its action on the standard basis vectors. R is just a normal rotation in the xy-plane, and completely fixes the z-axis, so using the formulas for a rotation in $2D$ one can compute where R maps e_1 and e_2. We then find $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

3. (a) Row reduce the augmented matrix: $\left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & a \\ 2 & 5 & 3 & 0 & a \\ 1 & 0 & -1 & 0 & -b \\ 0 & 1 & 1 & 0 & b \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & a \\ 0 & 1 & 1 & 2 & a \\ 0 & 0 & 0 & 10 & 4a-2b \\ 0 & 0 & 0 & 0 & -a+3b \end{array}\right)$.
   We see this system is consistent if and only if $a = 3b$. When this is true, the augmented matrix becomes $\left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 3b \\ 0 & 1 & 1 & 2 & 3b \\ 0 & 0 & 0 & 10 & 10b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$, which has infinitely many solutions.

   (b) Row reduce further: $\left(\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 3b \\ 0 & 1 & 1 & 2 & 3b \\ 0 & 0 & 0 & 10 & 10b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -b \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 0 & 1 & b \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$. This says $x = -b + z$, $y = b - z$, z is free and $t = b$. Any vector that solves the system is of the form $(x,y,z,t) = (-b+z, b-z, z, b) = z(1,-1,1,0) + (-b,b,0,b)$, for z arbitrary. Geometrically, this solution set is a line in $\mathbb{R}^4$.

4. (a) We can write this as a matrix equation of the form $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ 8 \end{pmatrix}$.

(b) A is invertible because $\text{rank}(A) = 3$. Row reduce A to see $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -4 & -7 \end{pmatrix} \to$ $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$, from which we can see A has full rank. To compute A^{-1}, we use the algorithm which says that RREF of the augmented matrix $[A|I_3]$ is given by $[I_3|A^{-1}]$. If you do the row reduction, you'll find $A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -6 & 7 & -2 \\ 4 & -4 & 1 \end{pmatrix}$. Since A is invertible, we know the system has a unique solution.

(c) The solution is given by $\vec{x} = A^{-1} \begin{pmatrix} 7 \\ 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ -16 \\ 12 \end{pmatrix}$. Geometrically, the solution set is just the point $(3, -16, 12)$.

5. Write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we want to solve $AB = BA$, i.e. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Doing the matrix multiplication, this says we want $\begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} = \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$. This gives us the system of equations $\begin{cases} a+2c = a+3b \\ b+2d = 2a+4b \\ 3a+4c = c+3d \\ 3b+4d = 2c+4d \end{cases}$. Solving the system says $d = a + \frac{3}{2}b$ and $c = \frac{3}{2}b$, with a and b free, so $B = \begin{pmatrix} a & b \\ \frac{3}{2}b & a + \frac{3}{2}b \end{pmatrix}$ for arbitrary a, b.

6. (a) Follow the order of composition. $A = DSRP$.

(b) T first projects $\mathbb{R}^2$ onto the line $y = \frac{1}{\sqrt{3}}x$, which is at an angle of $\pi/6$. This line is then rotated by an angle of $\pi/3$ to become the y-axis, which is then not changed by both the reflection and the scaling, so that $\text{Im}(T)$ is the y-axis. Any vector on the line orthogonal to $y = \frac{1}{\sqrt{3}}x$ through the origin is crushed to $\vec{0}$ by the projection, and any vector not on this line does not get mapped to $\vec{0}$ by T (which is easy to see geometrically). This says $\ker(T)$ is just the line orthogonal to $y = \frac{1}{\sqrt{3}}x$, which is the line $y = -\sqrt{3}x$.

(c) Using all the relevant formulas, we have $P = \begin{pmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{pmatrix}$, $R = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and $D = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. This gives $A = \begin{pmatrix} 0 & 0 \\ -3\sqrt{3}/2 & -3/2 \end{pmatrix}$. $\text{Im}(A)$ is spanned by the columns of A, which we see are just scalar multiples of each other, so it's really just spanned by one of them. This says $\text{Im}(A) = \text{Span}\{(0, -3/2)\} = \text{Span}\{(0, 1)\}$, which is just the y-axis. Similarly, we can read off the kernel to see that $\ker(A) = \{(x, y) : -3\sqrt{3}x/2 = 3y/2\} = \{(x, y) : y = -\sqrt{3}x\}$, which is the desired line.

7. (a) We have a set of 4 vectors in $\mathbb{R}^3$, so the matrix A whose columns are formed by these vectors is a 3×4 matrix. These vectors are linearly independent if and only if $\text{rank}(A) = 4$, but since $\text{rank}(A) \leq 3$ and $\text{rank}(A) \leq 4$ this says $\text{rank}(A) < 4$, which says they are linearly dependent.

(b) We want to solve $c_1(1, -1, 1) + c_2(1, 4, 5) + c_3(6, 8, 10) = (40, 33, 48)$ for constants c_1, c_2, c_3. This is equivalent to the matrix equation $\begin{pmatrix} 1 & 1 & 6 \\ -1 & 4 & 8 \\ 1 & 5 & 10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 40 \\ 33 \\ 48 \end{pmatrix}$. We can solve this by row reducing the augmented matrix: $\left(\begin{array}{ccc|c} 1 & 1 & 6 & 40 \\ -1 & 4 & 8 & 33 \\ 1 & 5 & 10 & 48 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 6 & 40 \\ 0 & 5 & 14 & 73 \\ 0 & 4 & 4 & 8 \end{array}\right) \to$ $\left(\begin{array}{ccc|c} 1 & 1 & 6 & 40 \\ 0 & 20 & 56 & 292 \\ 0 & -20 & -20 & -40 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 6 & 40 \\ 0 & 20 & 56 & 292 \\ 0 & 0 & 36 & 252 \end{array}\right)$. The last row gives $c_3 = 7$, which then says $20c_2 + 392 = 292$, so $c_2 = -5$ and $c_1 + 37 = 40$ says $c_1 = 3$.

8. Let A be the matrix of T. Then $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 2 & 3 & -5 \end{pmatrix}$. We find the image and kernel by row reduction: $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 2 & 3 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$. Since the first two columns have pivots, this says the image of A is spanned by the first two columns, i.e. $\mathrm{Im}(A) = \mathrm{Span}\{(1,2,2),(1,2,3)\}$. From this, we can also read off the kernel of A. Any vector in the kernel must satisfy $x+y-z=0$ and $y-3z=0$. This says $x=-y+z$ and $y=3z$ with z free, so vectors in the kernel are of the form $(x,y,z) = (-2z,3z,z) = z(-2,3,1)$. This says $\ker(A) = \mathrm{Span}\{(-2,3,1)\}$. From this, we see $\dim(\mathrm{Im}(A)) = 2$ and $\dim(\ker(A)) = 1$. Geometrically, $\mathrm{Im}(A)$ is a plane, and $\ker(A)$ is a line.

9. We need to compute $T(1,0)$ and $T(0,1)$, i.e. $A\vec{e_1}$ and $A\vec{e_2}$, which are the columns of A. Let $B = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Then $B\vec{e_1} = (3,1)$ and $B\vec{e_2} = (1,2)$ says $(AB)\vec{e_1} = A\begin{pmatrix} 3 \\ 1 \end{pmatrix} = T(3,1) = (2,3)$ and $(AB)\vec{e_2} = A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = T(1,2) = (9,1)$. This says $AB = \begin{pmatrix} 2 & 9 \\ 3 & 1 \end{pmatrix}$. Notice that B is invertible, with $B^{-1} = \begin{pmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{pmatrix}$, so that $A = \begin{pmatrix} 2 & 9 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 1 & 0 \end{pmatrix}$.

10. (a) We can choose $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 1 & -5 & 0 & 0 \end{pmatrix}$, for example.

(b) By how matrix multiplication works, vectors in $\ker(A)$ are necessarily orthogonal to the rows of A.

(c) We have $A^t = \begin{pmatrix} 1 & -1 & 3 & 1 \\ 2 & 3 & 4 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since $\mathrm{Im}(A^t)$ is spanned by the columns of A^t, which are the rows of A, any vector in $\ker(A)$ is therefore necessarily orthogonal to vectors in $\mathrm{Im}(A^t)$.

(d) Row reduce A^t: $\begin{pmatrix} 1 & -1 & 3 & 1 \\ 2 & 3 & 4 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 3 & 1 \\ 0 & 5 & -2 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 5 & 0 & 13 & -2 \\ 0 & 5 & -2 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. This says a vector (x,y,z,t) in $\ker(A^t)$ is of the form $(x,y,z,t) = (-13z/5+2t/5, 2z/5+7t/5, z, t) = z(-13/5, 2/5, 1, 0) + t(2/5, 7/5, 0, 1)$. From this, we can see that $\ker(A^t) = \mathrm{Span}\{(-13,2,5,0),(2,7,0,5)\}$. We then see that vectors in $\ker(A^t)$ are orthogonal to vectors in V (in fact, they are all such vectors).

Extra:

(a) Let $\vec{x} \in \ker(A)$. Then $A\vec{x} = \vec{0}$, so $A^2\vec{x} = A(A\vec{x}) = A(\vec{0}) = \vec{0}$. This says $\vec{x} \in \ker(A^2)$, so $\ker(A) \subseteq \ker(A^2)$. If $\vec{y} \in \mathrm{Im}(A^2)$, then $\vec{y} = A^2\vec{x}$ for some $\vec{x}$. But then $\vec{y} = A^2\vec{x} = A(A\vec{x})$, which says $\vec{y} \in \mathrm{Im}(A)$. Therefore, $\mathrm{Im}(A^2) \subseteq \mathrm{Im}(A)$.

(b) If $A = 0$ this is obvious, so assume $A \neq 0$. Then $\text{rank}(A) = 1$ or $\text{rank}(A) = 2$. If $\text{rank}(A) = 2$, then A is invertible, but if $A^3 = 0$ and A is invertible, multiplying by A^{-2} would say $A = 0$, which is not invertible. So actually, this cannot happen. Therefore, $\text{rank}(A) = 1$, so $\dim(\ker(A)) = 1$. By part (a), $\ker(A) \subset \ker(A^2)$ and $\text{Im}(A^2) \subset \text{Im}(A)$.

Suppose that $A^2 \neq 0$, then $\text{Im}(A^2)$ is a non-zero 1-dimensional subspace of $\text{Im}(A)$, i.e. $\text{Im}(A^2) = \text{Im}(A)$. We then have $\dim(\ker(A^2)) = 1$, and since $\dim(\ker(A)) = 1$ this says $\ker(A^2) = \ker(A)$. Finally, notice that if $\vec{y} \in \text{Im}(A)$, then $\vec{y} = A\vec{x} = A^2\vec{x'}$ for some vectors $\vec{x}, \vec{x'}$. Then $A\vec{y} = A^3\vec{x'} = \vec{0}$, so that $\vec{y} \in \ker(A)$. This then says that $\text{Im}(A) = \ker(A)$. Then for any $\vec{x}$, we have $A^2\vec{x} = A(A\vec{x}) = A(\vec{0}) = \vec{0}$, which says that $A^2 = 0$. This gives a contradiction, so that $A^2 = 0$ as desired.