Quiz 10 Solutions

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1. Let $q(x_1, x_2, x_3) = 4x_1x_2 + 4x_1x_3 + x_2^2 - x_3^2$.

(a) Find symmetric A such that $q = x^t Ax$.

(b) Orthogonally diagonalize $A = SDS^t$.

(c) Determine the definiteness of q.

(d) Maximize q subject to the constraint $\|x\| = 1$.

Solution:

(a) $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$.

(b) Check that $p_A(\lambda) = 9\lambda - \lambda^3 = \lambda(\lambda+3)(\lambda-3)$. One can compute that orthonormal bases for the eigenspaces are given by $E_0 = \text{Span}\{(1/\sqrt{5}, -2/\sqrt{5}, 2/\sqrt{5})\}$, $E_3 = \text{Span}\{(2/\sqrt{5}, 2/\sqrt{5}, 1/\sqrt{5})\}$, and $E_{-3} = \text{Span}\{(-2/\sqrt{5}, 1/\sqrt{5}, 2/\sqrt{5})\}$, so $S = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} & -2/\sqrt{5} \\ -2/\sqrt{5} & 2/\sqrt{5} & 1/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

(c) Since we have both positive and negative eigenvalues, q is indefinite.

(d) With respect the the orthonormal eigenbasis $\beta = \{v_1, v_2, v_3\}$ corresponding to the eigenvalues $0, 3, -3$ respectively, we want to maximize $q(c_1, c_2, c_3) = 3c_2^2 - 3c_3^2$, subject to the constraint $c_1^2 + c_2^2 + c_3^2 = 1$. We then have $3c_2^2 - 3c_3^2 \leq 3c_2^2 \leq 3$, since $c_2^2 \leq 1$. We see that $(c_1, c_2, c_3) = (0, 1, 0)$ actually attains this upper bound, so the maximal value is 3.