Quiz 8 Solutions

Tim Smits

1. Find the volume of the 3-parallelepiped defined by $v_1 = (1,0,0,0)$, $v_2 = (1,1,1,1)$, and $v_3 = (1,2,3,4)$.

> **Solution:** The volume is given by $\sqrt{\det(A^tA)}$, where $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{pmatrix}$. We compute $A^tA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{pmatrix}$, so $\det(A^tA) = 6$ gives a volume of $\sqrt{6}$.

2. Write down the set of eigenvalues for the following linear transformations:

(a) Reflection around the y-axis in $\mathbb{R}^2$.

(b) Reflection around the xy-plane in $\mathbb{R}^3$.

(c) Reflection around a plane $V = \{x \in \mathbb{R}^3 : n^tx = 0\}$ where n is normal to the plane.

(d) Orthogonal projection onto a plane $V = \{x \in \mathbb{R}^3 : n^tx = 0\}$ where n is normal to the plane.

> **Solution:**
>
> (a) The matrix of a reflection is orthogonal, and so in particular, has possible eigenvalues ± 1. Both possible values actually are eigenvalues (as seem by problem 3).
>
> (b) ± 1.
>
> (c) ± 1.
>
> (d) The matrix P of an orthogonal projection satisfies $P^2 = P$, so if λ is an eigenvalue it satisfies $\lambda^2 = \lambda$, i.e. $\lambda = 0, 1$. Both of these are actually eigenvalues by problem 3.

3. Find eigenvectors corresponding to the eigenvalues of the linear transformations in the above problem.

> **Solution:** Let A be the matrix of the relevant linear transformation in each part.
>
> (a) Let v be any non-zero vector on the y-axis, and w be any non-zero vector on the x-axis. Then $Av = v$ (because it's on the y-axis), and $Aw = -w$ (because it's on the x-axis).
>
> (b) Let v be any non-zero vector in the xy-plane and w be any vector orthogonal to the xy-plane. Then $Av = v$ (because v lies in the xy-plane) and $Aw = -w$ (because w is orthogonal to the xy-plane).
>
> (c) Let v be any non-zero vector in V and let w be any non-zero vector in $V^\perp$. We see that $Av = v$ (because v lies in the plane V), and $Aw = -w$ (because w is orthogonal to V).

(d) Let v be any non-zero vector in V and let w be any non-zero vector in $V^{\perp}$. Then $Av = v$ (because v is already in V), and $Aw = 0$ (because w is orthogonal to V).