Quiz 4 Solutions

Tim Smits

1. Let $\beta = \{(3,1),(5,2)\}$ be a basis of $\mathbb{R}^2$.

(a) Find P such that $[\vec{x}]_\beta = P\vec{x}$ for any $\vec{x} \in \mathbb{R}^2$.

(b) Find S such that $\vec{x} = S[\vec{x}]_\beta$ for any $x \in \mathbb{R}^2$.

Solution:

(a) The matrix P we are looking for is the change of basis matrix $S_{\mathcal{E}}^{\beta}$ where $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ is the standard basis of $\mathbb{R}^2$. We have $P = S_{\mathcal{E}}^{\beta} = (S_{\beta}^{\mathcal{E}})^{-1} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$.

(b) The matrix S we are looking for is the change of basis matrix $S_{\beta}^{\mathcal{E}}$. We have $S = S_{\beta}^{\mathcal{E}} = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with $T(3,1) = (6,2)$ and $T(5,2) = (15,6)$.

(a) Find A, the matrix of T.

(b) (Bonus): What is the rank of A?

Solution:

(a) Let $S = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$. The given conditions say $(AS)\vec{e}_1 = A\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ and $(AS)\vec{e}_2 = A\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \end{pmatrix}$. This says $AS = \begin{pmatrix} 6 & 15 \\ 2 & 6 \end{pmatrix}$, so that $A = \begin{pmatrix} 6 & 15 \\ 2 & 6 \end{pmatrix} S^{-1} = \begin{pmatrix} -3 & 15 \\ -2 & 8 \end{pmatrix}$.

Alternatively, let $\beta = \{(3,1),(5,2)\}$ and $\mathcal{E}$ be the standard basis. Then the matrix of T with respect to β-coordinates, A_β, is given by $A_\beta = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. The change of basis formula says $A = S_{\beta}^{\mathcal{E}} A_\beta S_{\mathcal{E}}^{\beta} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 15 \\ -2 & 8 \end{pmatrix}$.

(b) Many ways to see this; in particular, A is a product of invertible matrices, and therefore invertible, so $\text{rank}(A) = 2$.