Selected Solutions to Homwork 9

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**7.2.9** Find the eigenvalues of $A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{pmatrix}$ along with their algebraic multiplicities.

**Solution:** We have $A - \lambda I = \begin{pmatrix} 3-\lambda & -2 & 5 \\ 1 & -\lambda & 7 \\ 0 & 0 & 2-\lambda \end{pmatrix}$. Expanding the determinant along the last row, we have $p_A(\lambda) = \det(A - \lambda I) = (2-\lambda)(-\lambda(3-\lambda)+2) = -(\lambda-2)^2(\lambda-1)$. The eigenvalues are $\lambda = 2, 1$ with multiplicities 2 and 1 respectively.

**7.2.21** Prove that if an $n \times n$ matrix $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, that $\mathrm{Tr}(A) = \lambda_1 + \ldots + \lambda_n$.

**Solution:** The eigenvalues of $A$ are roots of the characteristic polynomial of $A$, so write $p_A(\lambda) = (\lambda_1 - \lambda)\ldots(\lambda_n - \lambda)$. Recall that the coefficient of $\lambda^{n-1}$ in $p_A(\lambda)$ is given by $(-1)^{n-1}\mathrm{Tr}(A)$. On the other hand, by directly expanding out the product, we see the coefficient of $\lambda^{n-1}$ in $p_A(\lambda)$ is given by $(-1)^{n-1}\lambda_1 + (-1)^{n-1}\lambda_2 + \ldots + (-1)^{n-1}\lambda_n = (-1)^{n-1}(\lambda_1 + \ldots + \lambda_n)$. Equating coefficients, this says $(-1)^{n-1}\mathrm{Tr}(A) = (-1)^{n-1}(\lambda_1 + \ldots + \lambda_n)$, so that $\mathrm{Tr}(A) = \lambda_1 + \ldots + \lambda_n$.

**7.2.30** Suppose that $A$ is an $n \times n$ matrix with positive entries such that the sum of each row of $A$ is equal to 1.

(a) Let $v$ be an eigenvector of $A$ with positive entries associated to the eigenvalue $\lambda$. Show that $\lambda \leq 1$.

(b) Now suppose that $v$ is an eigenvector with no restrictions on the entries with associated eigenvalue $\lambda$. Show that $|\lambda| \leq 1$.

(c) Show that $\lambda = -1$ is not an eigenvalue of $A$. Show that any eigenvector of eigenvalue 1 is of the form $(c, c, \ldots, c)$ for some $c \neq 0$.

**Solution:**

(a) Write $A = [a_{ij}]$ and suppose that $Av = \lambda v$. Let $v_i$ be the maximal entry of $v$. Then the $i$-th entry of $Av$ is given by $a_{i1}v_1 + \ldots + a_{in}v_n$, so we have $\lambda v_i = a_{i1}v_1 + \ldots + a_{in}v_n$. Since $v_i$ is maximal, we have $v_k \leq v_i$ for $1 \leq k \leq n$, so $a_{i1}v_1 + \ldots + a_{in}v_n \leq a_{i1}v_i + \ldots + a_{in}v_i = (a_{i1} + \ldots + a_{in})v_i = v_i$. This says $\lambda v_i \leq v_i$, and since $v_i > 0$, we find that $\lambda \leq 1$.

(b) Let $v_i$ be the entry of $v$ such that $|v_i|$ is maximal. Then $|v_i| \neq 0$, because $v$ is an eigenvector, so in particular, is non-zero. The $i$-th entry of $Av$ is given by $a_{i1}v_1 + \ldots + a_{in}v_n$, so we have $a_{i1}v_1 + \ldots + a_{in}v_n = \lambda v$. Taking absolute values, $|a_{i1}v_1 + \ldots + a_{in}v_n| = |\lambda||v_i|$. By the triangle inequality, $|a_{i1}v_1 + \ldots + a_{in}v_n| \leq |a_{i1}||v_1| + \ldots + |a_{in}||v_n| = a_{i1}|v_1| + \ldots + a_{in}|v_n| \leq a_{i1}|v_i| + \ldots + a_{in}|v_i| = (a_{i1} + \ldots + a_{in})|v_i| = |v_i|$. This gives $|\lambda||v_i| \leq |v_i|$, and since $|v_i| \neq 0$, this says $|\lambda| \leq 1$.

(c) First, notice that the rows of a matrix $A$ sum to 1 if and only if $e = (1, 1, \dots, 1)$ is an eigenvector of $A$ with eigenvalue 1. Then notice that if $A$ has rows that sum to 1, then so does $A^2$, because $A^2 e = A(Ae) = Ae = e$. Now suppose that $-1$ is an eigenvalue of $A$, so $Av = -v$ for some non-zero vector $v$. This says $A^2 v = -Av = v$, so that $v$ is an eigenvector of $A^2$ of eigenvalue 1. Since $A^2$ is a positive matrix whose rows sum to 1, assuming the second part of the statement of the problem, we conclude that $v = (c, c, \dots, c)$ for some $c \neq 0$. However, this would then say that $(c, c, \dots, c) = Av = -(c, c, \dots, c) = -v$, so that $c = 0$, which is a contradiction. Therefore, $-1$ is not an eigenvalue for $A$.

It remains to show that if $A$ is a positive matrix whose rows sum to 1, that every eigenvector of eigenvalue 1 is a multiple of $e$. Suppose that $Av = v$ for some $v$. Let $v_i$ be the maximal entry of $v$. Then looking at the $i$-th entry, we find $v_i = a_{i1}v_1 + \dots + a_{in}v_n \leq a_{i1}v_i + \dots a_{in}v_i = (a_{i1} + \dots + a_{ni})v_i = v_i$. In particular, this says $a_{i1}v_1 + \dots + a_{in}v_n = a_{i1}v_i + \dots a_{in}v_i = (a_{i1} + \dots + a_{ni})v_i$. Subtracting then gives $\sum_{k=1}^{n} a_{ik}(v_i - v_k) = 0$. Since $v_i$ is maximal, $v_i - v_k > 0$ for all $k$, and $a_{ik}$ are assumed to be positive. This then forces $v_i = v_k$ for $1 \leq k \leq n$, which says all entries of $v$ are the same. This then says that $v = (c, c, \dots, c)$ for some $c \neq 0$ as desired.

**7.2.33**

(a) Find the characteristic polynomial of $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$ for arbitrary $a, b, c$.

(b) Find a matrix with characteristic polynomial $-\lambda^3 + 17\lambda^2 - 5\lambda + \pi$.

**Solution:**

(a) We have $A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ a & b & c-\lambda \end{pmatrix}$, so expanding along the first row gives $\det(A - \lambda I) = -\lambda(-\lambda(c - \lambda) - b) + a = -\lambda^3 + c\lambda^2 + b\lambda + a$.

(b) Take the above matrix with $a = \pi$, $b = -5$ and $c = 17$.

**7.2.41** If $A$ is similar to $B$, prove that $\mathrm{Tr}(A) = \mathrm{Tr}(B)$.

**Solution:** Suppose that $A$ is similar to $B$, so that $A = SBS^{-1}$ for some invertible $S$. Then $\det(A - \lambda I) = \det(SBS^{-1} - \lambda I) = \det(S(B - \lambda I)S^{-1}) = \det(S)\det(B - \lambda I)\det(S^{-1}) = \det(B - \lambda I)$, so that $A$ and $B$ have the same characteristic polynomial. Since trace is the coefficient of the $\lambda^{n-1}$ term (up to sign), this says $\mathrm{Tr}(A) = \mathrm{Tr}(B)$.

**7.2.43** Are there $n \times n$ matrices $A, B$ with $AB - BA = I_n$?

**Solution:** No; if $AB - BA = I_n$, taking traces says $\mathrm{Tr}(AB - BA) = \mathrm{Tr}(I_n)$. We have $\mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = 0$, while $\mathrm{Tr}(I_n) = n$, so these cannot ever be equal.

**7.3.15** Determine if $A = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 0 & 1 \\ -4 & 0 & 3 \end{pmatrix}$ is diagonalizable, and if so, diagonalize $A$.

**Solution:** First, we find $p_A(\lambda)$. We have $A - \lambda I = \begin{pmatrix} -1-\lambda & 0 & 1 \\ -3 & -\lambda & 1 \\ -4 & 0 & 3-\lambda \end{pmatrix}$. Expanding the determinant along the second row, we find $p_A(\lambda) = -\lambda((3-\lambda)(-1-\lambda)+4) = -\lambda(\lambda-1)^2$. This says the eigenvalues of $A$ are $\lambda = 0, 1$ with algebraic multiplicities 0 and 2 respectively. To see if $A$ is diagonalizable, we need to check if the geometric multiplictity of 1 is equal to 2. We have $E_1 = \ker(A - I) = \ker \begin{pmatrix} -2 & 0 & 1 \\ -3 & -1 & 1 \\ -4 & 0 & 2 \end{pmatrix}$. We see this matrix has two linearly independent rows, so it has rank 2. Rank nullity says the kernel is therefore 1 dimensional, so $\dim(E_1) = 1 \neq 2$ says $A$ is not diagonalizable.

**7.3.21** Find a $2 \times 2$ matrix $A$ with $E_1 = \text{Span}\{(1,2)\}$ and $E_2 = \text{Span}\{(2,3)\}$. How many such matrices are there?

**Solution:** Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the conditions say that $Av = v$ and $Aw = 2w$, where $v = (1,2)$ and $w = (2,3)$, i.e. $A\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix}$. Solving gives $A = \begin{pmatrix} 1 & 4 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -2 \\ 6 & -2 \end{pmatrix}$. From the above, it's clear that $A$ is unique.

**7.3.25** What can you say about the geometric multiplicity of the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$ for arbitrary $a, b, c$?

**Solution:** Let $\lambda$ be an eigenvalue of $A$. Then $A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ a & b & c-\lambda \end{pmatrix}$. I claim that $\dim(E_\lambda) = 1$. To prove this, we give the following slick argument: we have $(A - \lambda I)^t = \begin{pmatrix} -\lambda & 0 & a \\ 1 & -\lambda & b \\ 0 & 1 & c-\lambda \end{pmatrix}$. Move the first row to the bottom (rank isn't changed by row operations!), and call the resulting matrix $B = \begin{pmatrix} 1 & -\lambda & b \\ 0 & 1 & c-\lambda \\ -\lambda & 0 & a \end{pmatrix}$. From the pivots in the first two rows, we see that $\text{rank}(B) \geq 2$. This then says $\text{rank}((A - \lambda I)^t) = \text{rank}(A - \lambda I) \geq 2$. Since $\lambda$ is an eigenvalue, $\dim(\ker(A - \lambda I)) \geq 1$ (you have an eigenvector!), and rank-nullity combined with the above rank inequality says $\dim(\ker(A - \lambda I)) \leq 1$, which combine to say that $\dim(E_\lambda) = 1$.

**7.3.37** Let $A$ be an $n \times n$ symmetric matrix.

(a) Show that if $v, w \in \mathbb{R}^n$ that $Av \cdot w = v \cdot Aw$.

(b) Prove that if $v$ and $w$ are eigenvectors for distinct eigenvalues, then $v$ is orthogonal to $w$.

**Solution:**

(a) $Av \cdot w = w^t Av$, and $v \cdot Aw = (Aw)^t v = w^t A^t v = w^t Av$ because $A$ is symmetric.

(b) Suppose $Av = \lambda_1 v$ and $Aw = \lambda_2 w$ for $\lambda_1 \neq \lambda_2$. Then $Av \cdot w = \lambda_1 v \cdot w$, and $v \cdot Aw = \lambda_2 v \cdot w$. By part $a$), this says $\lambda_1(v \cdot w) = \lambda_2(v \cdot w)$, so $(\lambda_1 - \lambda_2)(v \cdot w) = 0$. Since $\lambda_1 \neq \lambda_2$, this says $v \cdot w = 0$.

**7.3.47** For what constants $a, b, c$ is the matrix $A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ diagonalizable?

**Solution:** We have $p_A(\lambda) = (1-\lambda)^3$, so $A$ is diagonalizable if and only if $\dim(E_1) = 3$, i.e. $\ker(A - I) = \mathbb{R}^3$. This happens if and only if $A - I = 0$, so we require $a = b = c = 0$.

**7.3.53** Consider a $5 \times 5$ matrix $A$ and $v \in \mathbb{R}^5$. Suppose that $v, Av, A^2v$ are linearly independent, while $A^3 v = av + bAv + cA^2 v$ for some $a, b, c$. Expand $v, Av, A^2v$ to a basis $\beta = \{v, Av, A^2v, w_4, w_5\}$ of $\mathbb{R}^5$.

(a) Consider $B = [T]_\beta$ for $T(x) = Ax$. Write down the first three columns of $B$.

(b) Explain why $p_A(\lambda) = p_B(\lambda) = h(\lambda)(-\lambda^3 + c\lambda^2 + b\lambda + a)$ for some quadratic polynomial $h$.

(c) Explain why $p_A(A)v = 0$.

**Solution:**

(a) The first three columns of $B$ are given by $(0,1,0,0,0)$, $(0,0,1,0,0)$ and $(a,b,c,0,0)$.

(b) Write $B = \left(\begin{array}{ccc|cc} 0 & 0 & a & * & * \\ 1 & 0 & b & * & * \\ 0 & 1 & c & * & * \\ \hline 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{array}\right) = \left(\begin{array}{c|c} C & D \\ \hline 0 & E \end{array}\right)$ as a block partition. Then $p_B(\lambda) = \det(B - \lambda I_5) = \det(C - \lambda I_3)\det(E - \lambda I_2) = p_C(\lambda)p_E(\lambda)$. From 7.2.33, we know $p_C(\lambda) = -\lambda^3 + c\lambda^2 + b\lambda + a$, and $p_E(\lambda)$ is some quadratic polynomial $h(\lambda)$, since $E$ is a $2 \times 2$ matrix. Since $A$ and $B$ are similar, they have the same characteristic polynomial, so we find $p_A(\lambda) = p_B(\lambda) = (-\lambda^3 + c\lambda^2 + b\lambda + a)h(\lambda)$ as desired.

(c) We have $p_A(A) = (-A^3 + cA^2 + bA + aI_5)h(A) = h(A)(-A^3 + cA^2 + bA + aI_5)$, since both terms in the product only involve powers of $A$, so they commute. Then $p_A(A)v = h(A)(-A^3 + cA^2 + bA + aI_5)v = h(A)(-A^3v + cA^2v + bAv + av) = 0$, because $A^3v = av + bAv + cA^2v$.