Selected Solutions to Homwork 8

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6.3.13 Find the area of the parallelogram spanned by $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$.

Solution: The area is given by $\sqrt{\det(A^tA)}$ where $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$. We see that $A^tA = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$, so $\det(A^tA) = 20$ says the area is $\sqrt{20}$.

6.3.19 A basis v_1, v_2, v_3 of $\mathbb{R}^3$ is called *positively oriented* if v_1 encloses an acute angle with $v_2 \times v_3$. Show that v_1, v_2, v_3 is positively oriented if and only if $\det \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ is positive.

Solution: We have $\det \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} = v_1 \cdot (v_2 \times v_3) = \|v_1\|\|v_2 \times v_3\| \cos(\theta)$ where θ is the angle between v_1 and $v_2 \times v_3$. This determinant is positive if and only if $\cos(\theta) > 0$, and since $0 \leq 0 \leq \pi$, this happens if and only if $0 < \theta < \pi/2$, i.e. θ is an acute angle.

6.3.25 Find the classical adjoint of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, and use this to find A^{-1}.

Solution: The adjoint of A is given by $\begin{pmatrix} C_{11} & -C_{12} & C_{13} \\ -C_{21} & C_{22} & -C_{23} \\ C_{31} & -C_{32} & C_{33} \end{pmatrix}^t$, where C_{ij} is the (i, j)-th cofactor of A, i.e. the determinant of the matrix you get when you delete the i-th row and j-th column of A. If you do this (annoying) computation, you'll find that $\text{Adj}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$. We compute that $\det(A) = -1$, and we have $A^{-1} = \frac{1}{\det(A)}\text{Adj}(A)$, so $A^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$.

6.3.37 What is the relationship between $\text{Adj}(A)$ and $\text{Adj}(A^{-1})$?

Solution: By definition, we have $\text{Adj}(A) = \det(A)A^{-1}$ and $\text{Adj}(A^{-1}) = \det(A^{-1})A$. Since $\det(A^{-1}) = \frac{1}{\det(A)}$, this says $\text{Adj}(A^{-1}) = \frac{1}{\det(A)}A$. Notice that $\frac{1}{\det(A)}A$ is the inverse of $\text{Adj}(A)$, so that $\text{Adj}(A^{-1}) = (\text{Adj}(A))^{-1}$.

6.7 True or false: $\det(A + B) = \det(A) + \det(B)$ for any 5×5 matrices A and B.

Solution: False; take $A = B$. Then $det(A+B) = \det(2A) = 2^5 \det(A)$, which is obviously not $2\det(A)$.

6.19 If A is an $n \times n$ matrix, then $\det(AA^t) = \det(A^tA)$.

Solution: True; $\det(AA^t) = \det(A)\det(A^t) = \det(A^t)\det(A) = \det(A^tA)$.

6.37 If an $n \times n$ matrix A is invertible, then there must be an $(n-1) \times (n-1)$ submatrix of A (obtained by deleting a row and a column of A) that is invertible as well.

Solution: True; by the cofactor expansion of A, we have $\det(A) = \sum_{i,j}(-1)^{i+j}C_{ij}$, where C_{ij} is the ij-th cofactor of A. If A is invertible, then $\det(A) \neq 0$, so in particular, not every term in the sum can be 0. This says $C_{ij} \neq 0$ for some (i,j). Since C_{ij} is the determinant of the matrix you get when you delete the i-th row and j-th column of A, this says that A has an $(n-1) \times (n-1)$ submatrix with non-zero determinant, i.e. is invertible.

7.1.3 If v is an eigenvector of A with eigenvalue λ, is v an eigenvector of $A + 2I$?

Solution: Yes; $(A + 2I)v = Av + 2v = \lambda v + 2v = (\lambda + 2)v$. So v is an eigenvector with eigenvalue $\lambda + 2$.

7.1.13 Show that 4 is an eigenvalue of $A = \begin{pmatrix} -6 & 6 \\ -15 & 13 \end{pmatrix}$. Find all corresponding eigenvectors.

Solution: We need to show that $Av = 4v$ for some v, i.e. $(A - 4I)v = 0$ for some v. We compute that $A - 4I = \begin{pmatrix} -10 & 6 \\ -15 & 9 \end{pmatrix}$. We see that $\operatorname{rank}(A - 4I) = 1$ because the columns are multiples of each other, so that by rank-nullity, $\dim(\ker(A - 4I)) = 1$ says $A - 4I$ has a non-trivial kernel, i.e. that 4 is an eigenvalue of A. To find an eigenvector, we just need to find a single vector in the kernel (since it's 1-dimensional, any such eigenvector is necessarily just a scalar multiple). We see that $v = (3, 5)$ works.

7.1.19 Let P be the orthogonal projection onto a line L in $\mathbb{R}^3$. Is P diagonalizable? If so, find an eigenbasis.

Solution: Let v be a vector spanning the line L, and pick a basis $\{w_1, w_2\}$ of $L^\perp$. Then $\beta = \{v, w_1, w_2\}$ is a basis of $\mathbb{R}^3$. Geometrically, it's clear that $Pv = v$ because v lies on L to begin with, and that $Pw_1 = Pw_2 = 0$. This says that v is an eigenvector with eigenvalue 1, and w_1, w_2 are eigenvectors with eigenvalue 0. This says that $\{v, w_1, w_2\}$ is a basis of eigenvectors of P, i.e. an eigenbasis. Therefore, P is diagonalizable. With respect to this new basis β, we indeed see that $P_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is diagonal.

7.1.35 Show that similar matrices have the same eigenvalues.

Solution: Suppose that $A \sim B$, so that there is invertible S such that $A = S^{-1}BS$. If v is an eigenvalue of A with eigenvalue λ, then $Av = \lambda v$. This says $S^{-1}BSv = \lambda v$, so that $BSv = \lambda Sv$ after multiplying by S. This then says $B(Sv) = \lambda(Sv)$, so that Sv is an eigenvector of B with eigenvalue λ. This shows that any eigenvalue of A is an eigenvalue of B. Repeating the argument with the roles of A and B swapped says any eigenvalue of B is an eigenvalue of A, so that A and B have the same eigenvalues.

7.1.47 If v is an eigenvector of A, show that v is in the image of A or the kernel of A.

Solution: Suppose that $Av = \lambda v$. If $\lambda = 0$, this says $Av = 0$, so that $v \in \ker(A)$. If $\lambda \neq 0$, then $A(\frac{1}{\lambda}v) = v$, so that $v \in \operatorname{im}(A)$.