Selected Solutions to Homwork 7

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6.1.21 For what values of k is $A = \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix}$ invertible?

Solution: We have A is invertible if and only if $\det(A) \neq 0$. Expanding the determinant out along the first row, $\det(A) = k \det \begin{pmatrix} k & 1 \\ 1 & k \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ 1 & k \end{pmatrix} + \det \begin{pmatrix} 1 & k \\ 1 & 1 \end{pmatrix} = k(k^2-1) - (k-1) + (1-k) = k^3 - 3k + 2 = (k-1)^2(k+2)$. We then see that $\det(A) \neq 0$ for $k \neq 1, -2$.

6.1.29 Find the values of λ that make $A - \lambda I$ not invertible, where $A = \begin{pmatrix} 3 & 5 & 6 \\ 0 & 4 & 2 \\ 0 & 2 & 7 \end{pmatrix}$.

Solution: We have $A - \lambda I = \begin{pmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{pmatrix}$. Then $A - \lambda I$ is not invertible if and only if $\det(A - \lambda I) = 0$. Expanding the determinant out along the first column, $\det(A - \lambda I) = (3-\lambda) \det \begin{pmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{pmatrix} = (3-\lambda)[(4-\lambda)(7-\lambda) - 4)] = -(\lambda-3)^2(\lambda-8)$. Therefore, $\lambda = 3, 8$ are the values we seek.

6.1.43 If A is an $n \times n$ matrix, what is the relationship between $\det(A)$ and $\det(-A)$?

Solution: Write $A = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ where v_i are the rows of A. Then $-A = \begin{pmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix}$. The determinant is linear as a function of each fixed row. As a function of the first row, $\det(-A) = -\det \begin{pmatrix} v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix}$. As a function of the second row, $\det \begin{pmatrix} v_1 \\ -v_2 \\ \vdots \\ -v_n \end{pmatrix} = -\det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ -v_n \end{pmatrix}$, so $\det(-A) = (-1)^2 \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ -v_n \end{pmatrix}$. Continuing this logic, we eventually find $\det(-A) = (-1)^n \det(A)$.

6.1.57 What are the possible determinants for a permutation matrix?

Solution: Let A be an $n \times n$ permutation matrix, so that A has exactly one 1 in each row and column. Say the columns of A are $v_1, \ldots, v_n$, so by definition, $v_i = e_j$ for some j, where

e_j is a standard basis vector of $\mathbb{R}^n$. If T is the linear transformation associated to A, since $T(e_i) = v_i$, this says that T just permutes the set of basis vectors $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ into the set $\{v_1, \ldots, v_n\}$. We can define an inverse of T by defining $T^{-1}(v_i) = e_i$ (note that $\{v_1, \ldots, v_n\}$ is a basis, so this is all we need to define a linear transformation!), and then it's clear that $T(T^{-1}(v_i)) = v_i$ and $T^{-1}(T(e_i)) = e_i$. Furthermore, by construction, T^{-1} itself is just another permutation, so that A^{-1} is a permutation matrix. This says that A and A^{-1} are both matrices with *integer* entries, so by 6.2.37 this says $\det(A) = \pm 1$.

6.2.15 Let A be a 4×4 matrix with rows v_i. Suppose that $\det(A) = 8$. What is $\det \begin{pmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 + v_4 \end{pmatrix}$?

Solution: Perform row reduction: subtract row 3 from row 4, row 2 from row 3, and row 1 from row 2. We find $\det \begin{pmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 + v_4 \end{pmatrix} = \det(A) = 8$.

6.2.31 A *Vandermonde matrix* is an $n \times n$ matrix of the form $A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ a_0 & a_1 & \ldots & a_n \\ a_0^2 & a_1^2 & \ldots & a_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ a_0^n & a_1^n & \ldots & a_n^n \end{pmatrix}$. Vandermonde showed that $\det(A) = \prod_{i>j}(a_i - a_j)$.

(a) Verify this for $n = 1$.

(b) Suppose that the formula holds for any $n-1 \times n-1$ Vandermonde matrix. Define $f(t) = \det \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ a_0 & a_1 & \ldots & a_{n-1} & t \\ a_0^2 & a_1^2 & \ldots & a_{n-1}^2 & t^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_0^n & a_1^n & \ldots & a_{n-1}^n & t^n \end{pmatrix}$. Show that $f(t)$ is a degree n polynomial with roots $a_0, a_1, \ldots, a_{n-1}$. Let k be the coefficient of t^n in $f(t)$. Conclude that $f(t) = k(t - a_0) \ldots (t - a_{n-1})$ for some k. Plugging in a_{n-1}, deduce Vandermonde's formula.

Solution:

(a) We have $\det \begin{pmatrix} 1 & 1 \\ a_0 & a_1 \end{pmatrix} = a_1 - a_0$, so the formula holds.

(b) Expand the determinant of the matrix out along the last column, it's then easy to see that it must be a degree n polynomial in the variable t. Using the formula for the cofactor expansion, the coefficient k of t^n is given by $(-1)^{2n} \det(A_{nn}) = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ a_0 & a_1 & \ldots & a_{n-1} \\ a_0^2 & a_1^2 & \ldots & a_{n-1}^2 \\ \vdots & \vdots & \vdots & \vdots \\ a_0^{n-1} & a_1^{n-1} & \ldots & a_{n-1}^{n-1} \end{pmatrix} = \prod_{n-1 \geq i > j}(a_i - a_j)$ by assumption, since this is an $n-1 \times n-1$ Vandermonde matrix.

Now, $f(a_0) = f(a_1) = \ldots = f(a_{n-1}) = 0$, because the matrix will have a repeated column and therefore will not be invertible, and thus have 0 determinant. Since $f(t)$ is a degree n polynomial and we have found n roots of $f(t)$, this says $f(t) = k(t-a_0)\ldots(t-a_{n-1})$. If we plug in $t = a_n$, we find $\det(A) = k(a_n - a_0)\ldots(a_n - a_{n-1}) = \left(\prod_{n-1\geq i>j}(a_i - a_j)\right)(a_n - a_0)\ldots(a_n - a_{n-1}) = \prod_{n\geq i>j}(a_i - a_j)$. By induction, this proves Vandermonde's formula!

6.2.37 Consider an $n \times n$ matrix A such that both A and A^{-1} have integer entries. What are the possible values for $\det(A)$?

Solution: The key fact is that the determinant of a matrix A is a *polynomial* in the entries of the matrix A as a result of the cofactor expansion formula. In particular, since A and A^{-1} both have integer entries, $\det(A)$ and $\det(A^{-1})$ are just sums of products of integers, and therefore both integers. Since $\det(A)\det(A^{-1}) = \det(I) = 1$, this forces $\det(A) = \det(A^{-1}) = \pm 1$, since these are the only integer solutions to $xy = 1$.

6.2.41 Suppose A is an $n \times n$ skew-symmetric matrix, where n is odd. Show that A is not invertible.

Solution: Since A is skew-symmetric, we have $A = -A^t$. Then $\det(A) = \det(-A^t) = (-1)^n \det(A^t) = (-1)^n \det(A) = -\det(A)$ since n is odd. This says $\det(A) = 0$, so A is not invertible.

6.2.55 Let $D : \mathbb{R}^{n\times n} \to \mathbb{R}$ be a function such that D is linear in the rows, $D(B) = -D(A)$ if B is a matrix formed from swapping two rows of A, and $D(I) = 1$. Show that $D(A) = \det(A)$.

Solution: From the properties of D, we can deduce the following:

- $D(B) = \frac{1}{k}D(A)$ if B is obtained by dividing a row of A by k.
- $D(A) = 0$ if two rows of A are the same.
- If B is given by adding a multiple of a row of A to another, then $D(B) = D(A)$.

In particular, that says row reduction has the same effect on D that it has on the determinant. The same proof of the algorithm in the textbook goes through (since it only used these properties!), so we have $D(A) = (-1)^s k_1 \ldots k_r D(\text{rref}(A))$, where you swap rows s times and divide rows by scalars $k_1, \ldots, k_r$ while performing the row reduction. If A is invertible, then $\text{rref}(A) = I$, so $D(\text{rref}(A)) = 1$ says $D(A) = (-1)^s k_1 \ldots k_r = \det(A)$. If A is not invertible, then $\text{rref}(A)$ has a row of zeroes, so $D(\text{rref}(A)) = 0$ (again, the same proof that $\det(A) = 0$ if A has a row of zeroes still goes through!) which says $D(A) = 0 = \det(A)$. This shows that $D(A) = \det(A)$ as desired.

Since any matrix corresponds to a n-tuple of vectors $(v_1, \ldots, v_n)$ where $v_i \in \mathbb{R}^n$ and vice-versa by making the vectors v_i the rows of a matrix A (or columns), we may think of D as a function from $V^n \to \mathbb{R}$, (for us, we take $V = \mathbb{R}^n$), by saying $D(v_1, \ldots, v_n) = D\left(\begin{pmatrix} \text{-}v_1\text{-} \\ \vdots \\ \text{-}v_n\text{-} \end{pmatrix}\right)$.

Then the function $D(v_1, \ldots, v_n)$ is linear in each variable,

$D(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -D(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n)$, and $D(e_1, \ldots, e_n) = 1$. A function $V^n \to \mathbb{R}$ satisfying the first two properties is called an *alternating multi-linear n-form* on V. This problem shows that there is a unique alternating multi-linear n-form on $\mathbb{R}^n$ that maps the basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ to 1. In more advanced courses (typically at the graduate level) where you need linear algebra, this unique map is often times taken as the *definition* of the determinant!

6.2.56 Prove that $\det(AM) = \det(A)\det(M)$.

Solution: First suppose that M is not invertible. Then M has a non-trivial kernel. Say $x \in \ker(M)$. Then $(AM)x = A(Mx) = 0$, so AM is also not invertible. This says $\det(AM) = 0$, and since $\det(M) = 0$, we have $\det(A)\det(M) = 0$, so $\det(AM) = \det(A)\det(M)$ in this case.

Now suppose that M is invertible. Consider the function $D(A) = \frac{\det(AM)}{\det(M)}$. We will show that D has the properties listed in the previous problem, so that $D(A) = \det(A)$. We have $D(I) = \frac{\det(M)}{\det(M)} = 1$. By how matrix multiplication works, if we swap a row of A, it swaps a row of AM (write down A in terms of rows and M in terms of columns to easily compute the entries of AM if this is not clear!). Since swapping a row of AM negates $\det(AM)$, this says swapping a row of A negates the sign of $D(A)$. Finally, we just need to check that $D(A)$ is linear as a function of each row. Write $A = \begin{pmatrix} \text{-}r_1\text{-} \\ \vdots \\ \text{-}r_i\text{-} \\ \vdots \\ \text{-}r_n\text{-} \end{pmatrix}$ in terms of rows and $M = \begin{pmatrix} | & & | & & | \\ c_1 & \ldots & c_i & \ldots & c_n \\ | & & | & & | \end{pmatrix}$ in terms of columns. We have $\begin{pmatrix} \text{-}r_1\text{-} \\ \vdots \\ \text{-}kv+w\text{-} \\ \vdots \\ \text{-}r_n\text{-} \end{pmatrix} \begin{pmatrix} | & & | & & | \\ c_1 & \ldots & c_i & \ldots & c_n \\ | & & | & & | \end{pmatrix} = k \begin{pmatrix} \text{-}r_1\text{-} \\ \vdots \\ \text{-}v\text{-} \\ \vdots \\ \text{-}r_n\text{-} \end{pmatrix} \begin{pmatrix} | & & | & & | \\ c_1 & \ldots & c_i & \ldots & c_n \\ | & & | & & | \end{pmatrix} + \begin{pmatrix} \text{-}r_1\text{-} \\ \vdots \\ \text{-}w\text{-} \\ \vdots \\ \text{-}r_n\text{-} \end{pmatrix} \begin{pmatrix} | & & | & & | \\ c_1 & \ldots & c_i & \ldots & c_n \\ | & & | & & | \end{pmatrix} = kA_vM + A_wM$, where A_v and A_w are the matrices where you replace the i-th row with the vectors v and w. This says as a function of the i-th row, that $D(kv+w) = \frac{\det(kA_vM + A_wM)}{\det(M)} = \frac{k\det(A_vM) + \det(A_wM)}{\det(M)} = k\frac{\det(A_vM)}{\det(M)} + \frac{\det(A_wM)}{\det(M)} = kD(v) + D(w)$ since the determinant is linear as a function of the i-th row. It's obvious that $D(0) = 0$, so this proves that D is linear. By the previous problem, this says that $D(A) = \det(A)$, so that $\det(AM) = \det(A)\det(M)$ as desired.