Selected Solutions to Homwork 5

Tim Smits

5.1.17 Find a basis of $W^\perp$, where $W = \text{Span}\{(1,2,3,4),(5,6,7,8)\}$.

Solution: A vector $\vec{x} \in \mathbb{R}^n$ is an element of $W^\perp$ if and only if $\vec{x}\cdot(1,2,3,4) = 0$ and $\vec{x}\cdot(5,6,7,8) = 0$. Writing $\vec{x} = (a,b,c,d)$, this says we require $a+2b+3c+4d = 0$ and $5a+6b+7c+8d = 0$. Solving the system says $W^\perp = \text{Span}\{(1,-2,1,0),(2,-3,0,1)\}$, which is obviously linearly independent, and therefore a basis.

5.1.27 Find the orthogonal projection of $9\vec{e}_1$ onto the subspace V spanned by $(2,2,1,0)$ and $(-2,2,0,1)$.

Solution: The vectors $\vec{v}_1 = (2,2,1,0)$ and $\vec{v}_2 = (-2,2,0,1)$ are orthogonal, so normalize to make them orthonormal. We take $\vec{u}_1 = (2/3,2/3,1/3,0)$ and $\vec{u}_2 = (-2/3,2/3,0,1)$. Then $\text{Proj}_V(9\vec{e}_1) = (9\vec{e}_1\cdot\vec{u}_1)\vec{u}_1 + (9\vec{e}_1\cdot\vec{u}_2)\vec{u}_2 = 6\vec{u}_1 - 6\vec{u}_2 = (8,0,2,-2)$.

5.1.31 Consider orthonormal vectors $\vec{u}_1,\ldots,\vec{u}_m$ in $\mathbb{R}^n$, and for $\vec{x}\in\mathbb{R}^n$, set $p = (\vec{u}_1\cdot\vec{x})^2+\ldots+(\vec{u}_m\cdot\vec{x})^2$. What is the relationship between p and $\|\vec{x}\|^2$? When are they equal?

Solution: Let $V = \text{Span}\{\vec{u}_1,\ldots,\vec{u}_m\}$. Then notice that $\text{Proj}_V(\vec{x}) = (\vec{u}_1\cdot\vec{x})\vec{u}_1+\ldots+(\vec{u}_m\cdot\vec{x})\vec{u}_m$. Since the $\vec{u}_i$ are orthonormal, the Pythagorean theorem says $\|\text{Proj}_V(\vec{x})\|^2 = (\vec{u}_1\cdot\vec{x})^2+\ldots+(\vec{u}_m\cdot\vec{x})^2 = p$. We can write $\vec{x} = \text{Proj}_V(\vec{x}) + \vec{x}^\perp$, so again by Pythagorean theorem, $\|\vec{x}\|^2 = \|\text{Proj}_V(\vec{x})\|^2 + \|\vec{x}^\perp\|^2 = p + \|\vec{x}^\perp\|^2$. Since $\|\vec{x}^\perp\|^2 \geq 0$, in particular, this says $p \leq \|\vec{x}\|^2$. Equality holds if and only if $\|\vec{x}^\perp\|^2 = 0$, i.e. $\vec{x}^\perp = 0$ which is equivalent to saying that $\vec{x}\in V$.

5.1.39 Is there a line L in $\mathbb{R}^n$ and a vector $\vec{x}\in\mathbb{R}^n$ such that $\vec{x}\cdot\text{Proj}_L(\vec{x}) < 0$?

Solution: No; Let L be a line in $\mathbb{R}^n$ and write $L = \text{Span}\{\vec{u}\}$ for some unit vector $\vec{u}$. Then $\vec{x}\cdot\text{Proj}_L(\vec{x}) = \vec{x}\cdot((\vec{x}\cdot\vec{u})\vec{u}) = (\vec{x}\cdot\vec{u})^2 \geq 0$.

5.2.13 Perform Gram-Schmidt on the following vectors: $(1,1,1,1),(1,0,0,1),(0,2,1,-1)$.

Solution: Set $\vec{u}_1 = \vec{v}_1/\|\vec{v}_1\| = (1/2,1/2,1/2,1/2)$. Then $\vec{v}_2^\perp = \vec{v}_2 - (\vec{v}_2\cdot\vec{u}_1)\vec{u}_1 = \vec{v}_2 - \vec{u}_1 = (1/2,-1/2,-1/2,1/2)$. We actually have $\|\vec{v}_2^\perp\| = 1$, so $\vec{u}_2 = (1/2,-1/2,-1/2,1/2)$. We then have $\vec{v}_3^\perp = \vec{v}_3 - (\vec{v}_3\cdot\vec{u}_1)\vec{u}_1 - (\vec{v}_3\cdot\vec{u}_2)\vec{u}_2 = \vec{v}_3 - \vec{u}_1 + 2\vec{u}_2 = (1/2,1/2,-1/2,-1/2)$ which is again a unit vector, so $\vec{u}_3 = (1/2,1/2,-1/2,-1/2)$. This gives us the orthonormal set $\{(1/2,1/2,1/2,1/2),(1/2,-1/2,-1/2,1/2),(1/2,1/2,-1/2,-1/2)\}$.