Selected Solutions to Homwork 4

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3.3.69 Let V and W be two subspaces of $\mathbb{R}^n$. Prove that $\dim(V+W)+\dim(V\cap W)=\dim(V)+\dim(W)$.

Solution: Pick a basis $\vec{u}_1,\ldots,\vec{u}_m$ of $V\cap W$. Extend to a basis $\vec{u}_1,\ldots,\vec{u}_m,\vec{v}_1,\ldots,\vec{v}_p$ of V and a basis $\vec{u}_1,\ldots,\vec{u}_m,\vec{w}_1,\ldots,\vec{w}_q$ of W. The claim is that $\beta=\{\vec{u}_1,\ldots,\vec{u}_m,\vec{v}_1,\ldots,\vec{v}_p,\vec{w}_1,\ldots,\vec{w}_q\}$ is a basis of $V+W$, because then $\dim(V+W)=m+p+q$, $\dim(V\cap W)=m$, $\dim(V)=m+p$ and $\dim(W)=m+q$, so the result is easily deduced.

To prove this is a basis of $V+W$, we must show it's a linearly independent spanning set of $V+W$. First, we show β spans $V+W$: if $\vec{x}\in V+W$, then we may write $\vec{x}=\vec{v}+\vec{w}$ for some $\vec{v}\in V$ and some $\vec{w}\in W$. Then we may write $\vec{v}$ as a linear combination of the $\vec{u}_i$'s and the $\vec{v}_i$'s because they are a basis of V, and we may similarly write $\vec{w}$ as a linear combination of the $\vec{u}_i$'s and the $\vec{w}_i$'s because they are a basis of W. This says $\vec{x}$ may be written as a linear combination of the vectors in β, so they are a spanning set of $V+W$.

Next, we show that β is linearly independent. Suppose that $c_1\vec{u}_1+\ldots+c_m\vec{u}_m+d_1\vec{v}_1+\ldots+d_p\vec{v}_p+e_1\vec{w}_1+\ldots+e_q\vec{w}_q=\vec{0}$ for some constants c_i,d_i,e_i. Subtracting over says $c_1\vec{u}_1+\ldots+c_m\vec{u}_m+d_1\vec{v}_1+\ldots+d_p\vec{v}_p=-e_1\vec{w}_1-\ldots-e_q\vec{w}_q$. The left hand side is a vector in V, because it's a linear combination of vectors in V. On the other hand, the right hand side is a vector in W, because it's a linear combination of vectors in W. Therefore, each side is actually a vector in $V\cap W$. Then we may write $-e_1\vec{w}_1-\ldots-e_q\vec{w}_q$ as a linear combination of the vectors $\vec{u}_i$ (they are a basis of $V\cap W$!): $-e_1\vec{w}_1-\ldots-e_q\vec{w}_q=f_1\vec{u}_1+\ldots+f_m\vec{u}_m$, so that $e_1\vec{w}_1+\ldots+e_q\vec{w}_q+f_1\vec{u}_1+\ldots+f_m\vec{u}_m=\vec{0}$. Since the $\vec{u}_i$ and $\vec{w}_i$ are linearly independent (they are a basis of W!) this says $e_i=f_i=0$. In particular, our original equation becomes $c_1\vec{u}_1+\ldots+c_m\vec{u}_m+d_1\vec{v}_1+\ldots+d_p\vec{v}_p=\vec{0}$. Since the vectors $\vec{u}_i$ and $\vec{v}_i$ are linearly independent (they are a basis of V!), this says $c_i=d_i=0$. In particular, this says there is no non-trivial linear combination of these vectors equal to $\vec{0}$, they are linearly independent, and therefore a basis of $V+W$ as desired.

3.3.75 Let A be an $n\times n$ matrix. Show there are constants $c_0,\ldots,c_n$ not all 0 such that $c_0I_n+\ldots+c_nA^n$ is not invertible.

Solution: Pick a non-zero vector $\vec{v}\in\mathbb{R}^n$ and consider the set of vectors $\{\vec{v},A\vec{v},\ldots,A^n\vec{v}\}$. This is a set of $n+1$ vectors in $\mathbb{R}^n$, and since $\dim(\mathbb{R}^n)=n$ they must be linearly dependent. This says there are constants $c_0,\ldots,c_n$ such that $c_0\vec{v}+\ldots+c_nA^n\vec{v}=\vec{0}$, i.e. $(c_0I_n+\ldots+c_nA^n)\vec{v}=\vec{0}$. This says the matrix $c_0I_n+\ldots+A^n$ has a non-trivial kernel, so that it is not invertible.

3.3.79 Suppose that A is a nilpotent $n\times n$ matrix. Show that $A^n=0$.

Solution: Since A is nilpotent, there is m such that $A^m=0$. Let k be the smallest positive integer such that $A^k=0$ and $A^{k-1}\neq 0$. Pick $\vec{v}\in\mathbb{R}^n$ such that $A^{k-1}\vec{v}\neq\vec{0}$. By problem 78, the vectors $\{\vec{v},A\vec{v},\ldots,A^{k-1}\vec{v}\}$ are a linearly independent subset of k vectors in $\mathbb{R}^n$, and since $\dim(\mathbb{R}^n)=n$, in particular this says $k\leq n$. Then $A^n=A^kA^{n-k}=0$ as desired.

3.3.81 Prove that if k vectors span a k-dimensional subspace of $\mathbb{R}^n$, they are a basis.

Solution: Let V be a k-dimensional subspace of $\mathbb{R}^n$ such that $V = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$ for some vectors $\vec{v}_i$. To show these are a basis of V, we just need to check they are linearly independent. Suppose that $c_1\vec{v}_1 + \ldots + c_k\vec{v}_k = \vec{0}$. Let $T : \mathbb{R}^k \to \mathbb{R}^n$ be the linear transformation given by $T(\vec{x}) = A\vec{x}$ where $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_k \\ | & & | \end{pmatrix}$. By rank-nullity, $\text{rank}(A) + \dim(\ker(A)) = k$. Since $\text{rank}(A) = \dim(\text{im}(A))$ and $\text{im}(A) = V$ by construction, this says $\text{rank}(A) = k$ so $\dim(\ker(A)) = 0$, i.e. $\ker(A) = \{\vec{0}\}$. This says the vectors $\vec{v}_i$ are linearly independent, so we are done.

3.4.17 Find the coordinates of $\vec{x} = (1, 1, 1-1)$ with respect to the basis $\beta = \{(1, 0, 2, 0), (0, 1, 3, 0), (0, 0, 4, 1)\}$ of $V = \text{Span}\{(1, 0, 2, 0), (0, 1, 3, 0), (0, 0, 4, 1)\}$.

Solution: Note that $\vec{x} = \vec{v}_1 + \vec{v}_2 - \vec{v}_3$, so that $[\vec{x}]_\beta = (1, 1, -1)$.

3.4.29 Find the matrix of T with respect to the basis $\beta = \{(1, 1, 1), (1, 2, 3), (1, 3, 6)\}$ where $T(\vec{x}) = A\vec{x}$ for $A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{pmatrix}$,

Solution: Use the change of basis formula: $A_\beta = S^\beta_\mathcal{E} A S^\mathcal{E}_\beta$ where $\mathcal{E}$ is the standard basis of $\mathbb{R}^3$. By definition, $S^\mathcal{E}_\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}$ and $S^\beta_\mathcal{E} = (S^\mathcal{E}_\beta)^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$, so

$$A_\beta = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

3.4.57 Show that if A is the matrix of a reflection around a plane in $\mathbb{R}^3$, then A is similar to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Solution: Let S be the plane of reflection. Recall that $A = I_3 - 2P$, where P is the matrix of the orthogonal projection onto the line in the direction of a normal vector (which we will call $\vec{v}_3$) to S. Consider the basis $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$ where $\vec{v}_1$ and $\vec{v}_2$ are basis vectors of S and $\vec{v}_3$ is a vector on the line orthogonal to S (why is this a basis?). Then in β-coordinates, we have $A_\beta = I_3 - 2P_\beta$. To compute P_β, we see what the projection P does to the basis vectors $\vec{v}_i$. Since $\vec{v}_1$ and $\vec{v}_2$ are orthogonal to $\vec{v}_3$, $P\vec{v}_1 = P\vec{v}_2 = \vec{0}$, and since $\vec{v}_3$ is already on the line in the direction of $\vec{v}_3$, then $P\vec{v}_3 = \vec{v}_3$. That is, $P_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Plugging in gives $A_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which then says that A is similar to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

3.4.71 Suppose that A is similar to B, i.e. $B = S^{-1}AS$ for some S.

(a) Show that if $\vec{x} \in \ker(B)$, then $S\vec{x} \in \ker(A)$.

(b) Prove that $\dim(\ker(A)) = \dim(\ker(B))$.

Solution:

(a) Suppose $\vec{x} \in \ker(B)$, so $B\vec{x} = \vec{0}$. Then $\vec{0} = (SB)\vec{x} = (AS)\vec{x}$, so $A(S\vec{x}) = \vec{0}$ says $S\vec{x} \in \ker(A)$.

(b) The same argument shows that if $\vec{x} \in \ker(A)$, that $S^{-1}\vec{x} \in \ker(B)$. Pick a basis $\vec{v}_1, \dots, \vec{v}_k$ of $\ker(B)$ and a basis $\vec{w}_1, \dots, \vec{w}_\ell$ of $\ker(A)$. Then since S is invertible, the vectors $S\vec{v}_1, \dots, S\vec{v}_k$ are linearly independent vectors in $\ker(A)$, i.e. $k \leq \ell$. Reversing the argument says $S^{-1}\vec{w}_1, \dots, S^{-1}\vec{w}_\ell$ are linearly independent vectors in $\ker(B)$, i.e. $k \geq \ell$ so that $k = \ell$ as desired.

3.37 True or false: if V and W are subspace of $\mathbb{R}^n$, then $V \cup W$ is a subspace of $\mathbb{R}^n$.

Solution: False; Consider the two subspaces $V = \text{Span}\{\vec{e}_1\}$ and $W = \text{Span}\{\vec{e}_2\}$. Then $\vec{e}_1 \in V \cup W$ and $\vec{e}_2 \in W \cup W$ but $\vec{e}_1 + \vec{e}_2 \notin V \cup W$, so $V \cup W$ is not closed under addition, and therefore not a subspace.

3.39 If $\vec{v}_1, \dots, \vec{v}_n$ and $\vec{w}_1, \dots, \vec{w}_n$ are two bases of $\mathbb{R}^n$, there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\vec{v}_i) = \vec{w}_i$.

Solution: True; Set $\beta = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\gamma = \{\vec{w}_1, \dots, \vec{w}_n\}$. Then the condition $T(\vec{v}_i) = \vec{w}_i$ says that $A_\beta = \begin{pmatrix} | & & | \\ [\vec{w}_1]_\beta & \dots & [\vec{w}_n]_\beta \\ | & & | \end{pmatrix}$ where A_β is the matrix of T in β-coordinates (note: this also says that $A_\beta = S^\beta_\gamma$). One can then explicitly find A (in terms of the standard coordinates) using the change of basis formula: $A = S^{\mathcal{E}}_\beta A_\beta S^\beta_{\mathcal{E}}$.

More explicitly, $S^{\mathcal{E}}_\beta A_\beta$ is the matrix of the linear transformation T with respect to the bases β and $\mathcal{E}$ of $\mathbb{R}^n$, i.e. $S^{\mathcal{E}}_\beta A_\beta = A^{\mathcal{E}}_\beta = \begin{pmatrix} | & & | \\ [T(\vec{v}_1)]_{\mathcal{E}} & \dots & [T(\vec{v}_n)]_{\mathcal{E}} \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_n \\ | & & | \end{pmatrix}$, and $S^\beta_{\mathcal{E}} = (S^{\mathcal{E}}_\beta)^{-1} = \begin{pmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{pmatrix}^{-1}$, so we can explicitly write $A = \begin{pmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_n \\ | & & | \end{pmatrix} \begin{pmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{pmatrix}^{-1}$.

Note: this is a very useful (and important!) computation to be able to do. Make sure you understand how to do it!