Selected Solutions to Homwork 3

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2.4.5 Determine if $A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ is invertible, and if so, find its inverse.

Solution: Perform row reduction on A: $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. At this point we can stop, because we see that A does not have full rank, and therefore is not invertible.

2.4.77 Let $S = \begin{pmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{pmatrix}$ be invertible, and suppose $T(\vec{v}_i) = \vec{w}_i$ for some linear transformation T and vectors $\vec{w}_1, \dots, \vec{w}_m$. Set $B = \begin{pmatrix} | & & | \\ \vec{w}_1 & \dots & \vec{w}_m \\ | & & | \end{pmatrix}$. Compute the matrix A of T in terms of B and S.

Solution: Identifying S and B with their corresponding linear transformations, we have that $S(\vec{e}_i) = \vec{v}_i$ by definition of S, so that $(T \circ S)(\vec{e}_i) = T(\vec{v}_i) = \vec{w}_i = B(\vec{e}_i)$. This says $T \circ S = B$ as linear transformations, so as matrices, $AS = B$. Since S is invertible, we find $A = BS^{-1}$.

2.49 If $A^2 + 3A + 4I_3 = 0$, for a 3×3 matrix A, then A is invertible.

Solution: This is true: we see $A^2 + 3A = -4I_3$, so factoring says $A(A + 3I_3) = -4I_3$. Dividing then says $A(-\frac{1}{4}A - \frac{3}{4}I_3) = I_3$. It's also easy to check that $(-\frac{1}{4}A - \frac{3}{4}I_3)A = I_3$, so that A is invertible with $A^{-1} = -\frac{1}{4}A - \frac{3}{4}I_3$.

2.53 Is there a 10×10 matrix with 92 ones among its entries?

Solution: The answer is no. If A is such a matrix, then there are at least two rows of A that do not contain any 1's. This says A has two identical rows, namely a row of all 1's, and therefore has a 0 row upon row reduction. This says A does not have full rank, so A is not invertible.

3.1.11 Find vectors that span the kernel of $A = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{pmatrix}$.

Solution: The kernel of A is the same as the solution set to $A\vec{x} = \vec{0}$, which we know we can find by performing row reduction on the augmented matrix $\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & 0 \end{array}\right)$. We do this: $\left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \to$ $\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$. Reading off the equations, $x_1 + 2x_3 = 0$, $x_2 - 3x_3 = 0$, $x_4 = 0$ and x_3 is free. This says a vector (x_1, x_2, x_3, x_4) in the kernel is of the form $(x_1, x_2, x_3, x_4) = (-2x_3, 3x_3, x_3, 0) = x_3(-2, 3, 1, 0)$ for arbitrary x_3. Therefore, $\ker(A) = \text{Span}\{(-2, 3, 1, 0)\}$.

3.1.31 Give an example of a matrix A such that $\text{im}(A)$ is the plane orthogonal to the vector $(1, 3, 2)$.

Solution: Recall that $\text{im}(A)$ is spanned by the columns of A, so we just need a matrix whose columns span the desired plane, which is given by the equation $x + 3y + 2z = 0$. Any vector (x, y, z) such that $x + 3y + 2z = 0$ satisfies $x = -3y - 2z$, with y and z free. So $(x, y, z) = (-3y - 2z, y, z) = y(-3, 1, 0) + z(-2, 0, 1)$ says the vectors $(-3, 1, 0)$ and $(-2, 0, 1)$ span this plane. Therefore, $A = \begin{pmatrix} -3 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is such an example.

3.1.33 Give an example of a linear transformations whose kernel is the plane $x + 2y + 3z = 0$.

Solution: Let $\vec{v} = (1, 2, 3)$, and let $A\vec{x} = \vec{v} \cdot \vec{x}$. Then the kernel of A is the set of vectors such that $\vec{v} \cdot \vec{x} = 0$, i.e. with $\vec{x} = (x, y, z)$, all vectors with $x + 2y + 3z = 0$. The columns of the matrix A can be determined by computing $A\vec{e}_i$, which are given by $A\vec{e}_1 = 1$, $A\vec{e}_2 = 2$, and $A\vec{e}_3 = 3$ so that $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ is a matrix with the desired property.

3.2.1 Is $W = \{(x, y, z) : x + y + z = 1\}$ a subspace of $\mathbb{R}^3$?

Solution: No. W is not a subspace of $\mathbb{R}^3$ because the zero vector is not an element of W.

3.2.41 Let A be an $m \times n$ matrix and B be a $n \times m$ matrix such that $AB = I_m$ and $n \neq m$. Are the columns of B linearly independent? What about the columns of A?

Solution: If $B\vec{x} = \vec{0}$, then $\vec{0} = A\vec{0} = A(B\vec{x}) = (AB)\vec{x} = I_m\vec{x} = \vec{x}$. This then says the columns of B are linearly independent. From this, we conclude that $\text{rank}(B) = m$, and in particular, this tells us that $m \leq n$. Since $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$, we deduce that $\text{rank}(A) < n$, i.e. A does not have full rank so it's columns are linearly dependent.

3.2.45 Are the columns of an invertible matrtix linearly independent?

Solution: Yes. Let $A = \begin{pmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{pmatrix}$ be an invertible $n \times n$ matrix. By definition, this says the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$. Write $\vec{x} = (c_1, \dots, c_n)$. By definition of matrix multiplication, $A\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$. This then says that if $c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{0}$, that all $c_i = 0$, which is precisely what it means for the columns of A to be linearly independent.

3.2.47 Consider linearly independent vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $\mathbb{R}^4$. Compute RREF(A) where $A = \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$.

Solution: Since $\vec{v}_i$ are linearly independent, the only solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ is $(c_1, c_2, c_3) = (0, 0, 0)$. That is to say, the equation $A\vec{x} = \vec{0}$ has only a trivial solution. This says that A must have full rank, so that RREF$(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

3.3.21 Compute RREF(A), and find a basis of im(A) and ker(A) where $A = \begin{pmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{pmatrix}$.

Solution: First, we do row reduction: $\begin{pmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 9 \\ 0 & -7 & -28 \\ 0 & -15 & -60 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$. Two first two columns are the pivot columns of RREF(A), so the first two columns of A span the image of A. This says im$(A) = \text{Span}\{(1, 4, 7), (3, 5, 6)\}$. Next, to find the kernel of A, we solve $A\vec{x} = \vec{0}$, which we can find from the augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. This says $x - 3z = 0$, $y + 4z = 0$, and z is free, with $\vec{x} = (x, y, z)$. This says $(x, y, z) = (3z, -4z, z) = z(3, -4, 1)$, so that $\ker(A) = \text{Span}\{(3, -4, 1)\}$.

3.3.27 Consider the vectors $(1, 1, 1, 1), (1, -1, 1, -1), (1, 2, 4, 8), (1, -2, 4, -8) \in \mathbb{R}^4$. Do they form a basis?

Solution: These four vectors form a basis of $\mathbb{R}^4$ if they are linearly independent. Saying they are linearly independent says that the matrix A whose columns are formed by these vectors is invertible, or equivalently, has full rank. We can check this via row reduction: $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -2 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 8 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{pmatrix}$. We then conclude that A has full rank, so that these vectors form a basis of $\mathbb{R}^4$.

3.3.29 Find a basis of the subspace of $\mathbb{R}^3$ defined by $2x + 3y + z = 0$.

Solution: Similarly to a previous problem, we can recognize the plane $2x + 3y + z = 0$ as the kernel of the matrix $A = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$. By the rank-nullity theorem, $\dim(\ker(A)) + \text{rank}(A) = 3$. Since $\text{rank}(A) = 1$, this says $\dim(\ker(A)) = 2$. Therefore, any two linearly independent vectors that lie in the plane must be a basis of the plane. Observe that $(1, 0, -2)$ and $(3, -2, 0)$ are two such linearly independent vectors, so that a basis is given by $\{(1, 0, -2), (3, -2, 0)\}$.

3.3.35 Let $\vec{v} \in \mathbb{R}^n$ be a non-zero vector. What is the dimension of the space of all vectors in $\mathbb{R}^n$ orthogonal to $\vec{v}$?

Solution: A vector $\vec{x}$ orthogonal to $\vec{v}$ satsifies $\vec{v} \cdot \vec{x} = 0$, so we can recognize this space as the kernel of the linear transformation $T(\vec{x}) = \vec{v} \cdot \vec{x}$. By rank-nullity, we have $\text{rank}(T) + \dim(\ker(T)) = n$. As $\text{im}(T) \subset \mathbb{R}$, we have $\text{rank}(T) = 0$ or $\text{rank}(T) = 1$. Since $\vec{v} \neq \vec{0}$, T is not the zero transformation, i.e. $\text{rank}(T) = 1$. This says $\dim(\ker(T)) = n - 1$.