Selected Solutions to Homwork 2

Tim Smits

2.1.5 Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T((1,0,0)) = (7,11)$, $T((0,1,0)) = (6,9)$ and $T((0,0,1)) = (-13,17)$. Find the matrix A of T.

Solution: Let $\vec{e_i}$ denote the vector with the i-th component equal to 1 and 0 elsewhere. Recall that $A\vec{e_i}$ returns the i-th column of the matrix A by homework 1. The given information says $A\vec{e_1} = (7,11)$, $A\vec{e_2} = (6,9)$ and $A\vec{e_3} = (-13,17)$. This then tells us that $A = \begin{pmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{pmatrix}$.

2.1.37 Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that $\vec{v}$ and $\vec{w}$ are two arbitrary vectors in $\mathbb{R}^2$ and that $\vec{x}$ is a third vector whose endpoint is on the line segment connecting the endpoints of $\vec{v}$ and $\vec{w}$. Is the endpoint of the vector $T(\vec{x})$ necessarily on the line segment connecting the endpoints of $T(\vec{v})$ and $T(\vec{w})$?

Solution: The answer is yes. Since the endpoint of $\vec{x}$ lies on the line segment between $\vec{v}$ and $\vec{w}$, we may write $\vec{x} = k\vec{v} + (1-k)\vec{w}$ for some $0 \le k \le 1$. Then $T(\vec{x}) = T(k\vec{v} + (1-k)\vec{w}) = kT(\vec{v}) + (1-k)T(\vec{w})$ because T is a linear transformation. This says the endpoint of $T(\vec{x})$ lies on the line segment between $T(\vec{v})$ and $T(\vec{w})$.

2.2.7 Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of $(2,1,2)$. Find the reflection of the vector $(1,1,1)$ around the line L.

Solution: A unit vector in the direction of L is given by $\vec{u} = (2/3, 1/3, 2/3)$. Then the projection of $\vec{v} = (1,1,1)$ onto L is given by $\text{Proj}_L(\vec{v}) = (\vec{v}\cdot\vec{u})\vec{u} = 5/3(2/3,1/3,2/3) = (10/9, 5/9, 10/9)$. We then have $\text{Ref}_L(\vec{v}) = 2\text{Proj}_L(\vec{v}) - \vec{v} = (20/9, 10/9, 20/9) - (1,1,1) = (11/9, 1/9, 11/9)$.

2.2.31 Find a non-zero 3×3 matrix A such that $A\vec{x}$ is orthogonal to $(1,2,3)$ for all $\vec{x} \in \mathbb{R}^3$.

Solution: Any vector (x,y,z) orthogonal to $(1,2,3)$ lies in the plane $x + 2y + 3z = 0$. Therefore, if we let A be the matrix of the orthogonal projection onto the plane $x + 2y + 3z = 0$, it will have the desired property. Recall that the projection onto a plane S is given by $\text{Proj}_S(\vec{x}) = \vec{x} - \text{Proj}_{\vec{n}}(\vec{x})$, where $\vec{n}$ is a normal vector to S. In our case, take $\vec{n} = (1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$, so that $\text{Proj}_{\vec{n}}(\vec{x}) = (\vec{n}\cdot\vec{x})\vec{n}$. To find the matrix of this linear transformation, we just compute the action on the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. We see that the matrix P of $\text{Proj}_{\vec{n}}(\vec{x})$ is given by $P = \begin{pmatrix} 1/14 & 1/7 & 3/14 \\ 1/7 & 2/7 & 3/7 \\ 3/14 & 3/7 & 9/14 \end{pmatrix}$, so from $\text{Proj}_S(\vec{x}) = \vec{x} - \text{Proj}_{\vec{n}}(\vec{x})$ we find $A = I - P = \begin{pmatrix} 13/14 & -1/7 & -3/14 \\ -1/7 & 5/7 & -3/7 \\ -3/14 & -3/7 & 5/14 \end{pmatrix}$ is a matrix with the desired property.

Alternatively, one can construct a matrix A as follows. The transformation $T(\vec{x}) = A\vec{x}$ is determined entirely what it does to the vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$. This is because any vector $\vec{x} = (x,y,z)$

can be written as $(x, y, z) = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$, so by linearity $T(\vec{x}) = xT(\vec{e}_1) + yT(\vec{e}_2) + zT(\vec{e}_3)$. In particular, if $T(\vec{e}_i)$ is orthogonal to $(1, 2, 3)$ for all i, this says $T(\vec{x})$, and therefore $A\vec{x}$, is orthogonal to $(1, 2, 3)$ for all i. Since $T(\vec{e}_i) = A\vec{e}_i$ is just the i-th column of A, any matrix whose columns are orthogonal to $(1, 2, 3)$ will work. Notice that the vector $(2, -1, 0)$ is orthogonal to $(1, 2, 3)$, so the matrix $A = \begin{pmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ has the desired property.

2.2.47 Let $T(\vec{x}) = A\vec{x}$ be a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^2$. Define $f(t) = T((\cos(t), \sin(t)) \cdot T((-\sin(t), \cos(t))$.

(a) Show that $f(t)$ is continuous.

(b) Show that $f(\pi/2) = -f(0)$.

(c) Deduce that there is some $c \in [0, \pi/2]$ such that $f(c) = 0$.

Solution:

(a) Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $f(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$ $= \begin{pmatrix} a\cos(t) + b\sin(t) \\ c\cos(t) + d\sin(t) \end{pmatrix} \cdot \begin{pmatrix} -a\sin(t) + b\cos(t) \\ -c\sin(t) + d\cos(t) \end{pmatrix} = (a\cos(t)+b\sin(t))(-a\sin(t)+b\cos(t))+$ $(c\cos(t) + d\sin(t))(-c\sin(t) + d\cos(t))$. Since sums and products of continuous functions are continuous, this says $f(t)$ is continuous.

(b) We see that $f(\pi/2) = -ab - cd$ and $f(0) = ab + cd$, so that $f(\pi/2) = -f(0)$.

(c) Since $f(0)$ and $f(\pi/2)$ have opposite signs, in particular one is positive and one is negative (or they are both 0). By part *a)* f is continuous, so by the intermediate value theorem, there is some $c \in [0, \pi/2]$ such that $f(c) = 0$.

Note: the function $f(t)$ is the dot product of $(T \circ R_t)(\vec{e}_1)$ and $(T \circ R_t)(\vec{e}_2)$, where R_t is a counterclockwise rotation by t. As mentioned in the full problem statement, this problem proves that there are orthogonal unit vectors $\vec{u}_1, \vec{u}_2$ such that $T(\vec{u}_1)$ and $T(\vec{u}_2)$ are orthogonal. The idea of the proof is to start with the standard orthogonal unit vectors $\vec{e}_1$ and $\vec{e}_2$, rotate them both by some angle t (and by varying t between 0 and 2π, this will hit all possible pairs of orthogonal vectors), and then see if these new vectors are orthogonal after applying T, which can be done by computing a dot product. This is where the otherwise mysterious function $f(t)$ is coming from.

2.3.47 Find a 2×2 matrix A such that $A^3 = A$ and all entries of A are non-zero.

Solution: We turn to geometry to find an example. If T is a linear transformation corresponding to an orthogonal projection, then applying T twice is the same as just applying T once; i.e. projecting a projected vector doesn't give new information. Algebraically, if A is the matrix of T, this says $A^2 = A$, so that $A^3 = A^2 = A$. Let L be the line in the direction of your favorite arbitrary vector with non-zero entries. Mine is $(3, 4)$. Then $\vec{u} = (3/5, 4/5)$ is a unit vector in the direction of L, and the matrix of orthogonal projection onto L is given by $A = \begin{pmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{pmatrix}$ which has the desired property.