Selected Solutions to Homwork 1

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1.1.19 Consider the linear system

$$\begin{cases} x+y-z=-2 \\ 3x-5y+13z=18 \\ x-2y+5z=k \end{cases}$$

where k is an arbitrary number.

(a) For which values of k does the system have one or infinitely many solutions?

(b) For each value of k you found in part (a), how many solutions does the system have?

(c) Find all solutions for each value of k.

Solution:

(a) Translate the system of equations into the augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 3 & -5 & 13 & 18 \\ 1 & -2 & 5 & k \end{array}\right)$. Perform elimination: $\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 3 & -5 & 13 & 18 \\ 1 & -2 & 5 & k \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -8 & 16 & 24 \\ 0 & -3 & 6 & k+2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & k-7 \end{array}\right)$. For the system to be consistent, we require $k=7$.

(b) When $k=7$, the augmented matrix has a row of 0's. This says we have two relevant equations involving 3 variables, so there must be a free variable. This forces there to be infinitely many solutions.

(c) When $k=7$, we have seen the augmented system reduces to $\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Continuing elimination: $\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Translating back, this says $x=1-z$ and $y=-3+2z$. If we set $z=t$ as a free parameter, the solutions are given by $(x,y,z)=(1-t,-3+2t,t)=(1,-3,0)+t(-1,2,1)$.

1.2.18 Determine which of the following matrices are in RREF

(a) $\begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \end{pmatrix}$

Solution:

(a) No; the third column violates condition 2.

(b) Yes.

(c) No; the third column violates condition 3.

(d) Yes.

1.2.37 Find all vectors in $\mathbb{R}^4$ perpendicular to the three vectors $(1,1,1,1), (1,2,3,4), (1,9,9,7)$.

Solution: Let (a,b,c,d) an arbitrary vector in $\mathbb{R}^4$. Recall that two vectors $\vec{x}$ and $\vec{y}$ are perpendicular if $\vec{x}\cdot\vec{y}=0$. So we require that $(a,b,c,d)\cdot(1,1,1,1)=0$, $(a,b,c,d)\cdot(1,2,3,4)=0$ and $(a,b,c,d)\cdot(1,9,9,7)=0$. Algebraically, this translate into the system of equations

$$\begin{cases} a+b+c+d=0 \\ a+2b+3c+4d=0 \\ a+9b+9c+7d=0 \end{cases}$$

which we can then translate into the augmented matrix $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 9 & 9 & 7 & 0 \end{array}\right)$. Performing elimination, $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 9 & 9 & 7 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 8 & 8 & 6 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & -8 & -18 & 0 \end{array}\right) \to$ $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 9 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 4 & 9 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 4 & 0 & -4 & -8 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ 0 & 0 & 4 & 9 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 4 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 4 & 9 & 0 \end{array}\right) \to$ $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 1 & 9/4 & 0 \end{array}\right)$. This says $a=-d/4$ $b=3d/2$, and $c=-9d/4$, where d is a free variable. Setting $d=t$ as a free parameter, the solution set is given by $(a,b,c,d)=t(-1/4,3/2,9/4,1)$, which we recognize as a line in $\mathbb{R}^4$.

1.3.23 Consider a linear system of 4 equations with 3 variables. Suppose this system has a unique solution. What does the RREF of the coefficent matrix A look like?

Solution: A is a 4×3 matrix, and recall that #free variables $= m - \operatorname{rank}(A)$. If the system has a unique solution, we have no free variables, i.e. $\operatorname{rank}(A)=3$. This then says $\operatorname{RREF}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

1.3.47 A linear system of the form $A\vec{x}=\vec{0}$ is called *homogeneous*. Prove the following:

(a) All homogeneous systems are consistent.

(b) A homogeneous system with fewer equations than unknowns has infinitely many solutions.

(c) If $\vec{x_1}$ and $\vec{x_2}$ are solutions to $A\vec{x} = \vec{0}$, then $\vec{x_1} + \vec{x_2}$ is also a solution.

(d) If $A\vec{x} = \vec{0}$ and k is an arbitrary constant, $k\vec{x}$ is a solution to $A\vec{x} = \vec{0}$.

Solution:

(a) It's not possible to get a row of the form $\left(\begin{array}{cccc|c} 0 & 0 & \dots & 0 & * \end{array}\right)$ where the last entry is non-zero in the augmented matrix, so the system is consistent.

(b) If we have n equations with m variables with $n < m$, then A is an $n \times m$ matrix. From $\text{rank}(A) \leq n$ and $\text{rank}(A) \leq m$, in particular this says $\text{rank}(A) < m$, so #free variables $= m - \text{rank}(A) > 0$ forces there to be infinitely many solutions.

(c) We have $A(\vec{x_1} + \vec{x_2}) = A\vec{x_1} + A\vec{x_2} = \vec{0} + \vec{0} = \vec{0}$ by properties of matrices.

(d) We have $A(k\vec{x}) = k(A\vec{x}) = k\vec{0} = \vec{0}$ by properties of matrices.

1.3.59 For which values of c and d is the vector $(5, 7, c, d)$ a linear combination of $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$?

Solution: Saying that $(5, 7, c, d)$ is a linear combination of $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$ is the same as saying there are constants c_1 and c_2 such that $c_1(1, 1, 1, 1) + c_2(1, 2, 3, 4) = (5, 7, c, d)$. As a matrix equation, this is equivalent to saying $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ c \\ d \end{bmatrix}$. Look at the augmented system $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & c \\ 1 & 4 & d \end{array}\right)$. Perfoming elimination: $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & c \\ 1 & 4 & d \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & c-5 \\ 0 & 3 & d-5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & c-9 \\ 0 & 0 & d-11 \end{array}\right)$

In particular, we need the system to be consistent, so we need $c = 9$ and $d = 11$.