MULTIVARIATE TAYLOR POLYNOMIALS

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1. Introduction

For a function $f : \mathbb{R}^2 \to \mathbb{R}$, the tangent plane

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

provides a "good" approximation to the function $f(x, y)$ at points near (x_0, y_0), analogous to how in one dimension, a function $f : \mathbb{R} \to \mathbb{R}$ may be well approximated near a point x_0 by the tangent line

$$y - f(x_0) = f'(x_0)(x - x_0).$$

One of the core tools of calculus is the Taylor polynomial: for any $n \geq 1$, a smooth (i.e. infinitely differentiable) function $f : \mathbb{R} \to \mathbb{R}$ may be well-approximated at x_0 by the n-th order Taylor polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Taylor's theorem, one of the major results of single variable calculus, quantifies how "good" this approximation is:

Theorem 1.1 (Taylor)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, and let $x_0 \in \mathbb{R}$. Then for any $n \geq 1$, there exists a function $h_n(x)$ such that*

$$f(x) = P_n(x) + h_n(x)(x - x_0)^n$$

and

$$\lim_{x \to x_0} h_n(x) = 0.$$

The function $h_n(x)$ controls the error term $R_n(x) = f(x) - P_n(x)$, and the conditions on $h_n(x)$ say that the error tends to 0 faster than $(x - x_0)^n$ as $x \to x_0$. The purpose of this handout is to investigate the multivariate analogue of the Taylor polynomial, and a higher dimensional version of Taylor's theorem.

2. Definitions and Examples

Perhaps unsurprisingly, the correct way to generalize a Talyor polynomial to multiple dimensions is to include all the partial derivatives of a given degree. For the sake of concreteness, we'll restrict our attention to functions $f : \mathbb{R}^2 \to \mathbb{R}$. However, everything we do will naturally generalize to higher dimensions.

Definition 2.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be smooth. For $P = (x_0, y_0)$ the *n-th order Taylor polynomial* $P_n(x, y)$ is defined by

$$P_n(x, y) = \sum_{k=0}^{n} \sum_{i+j=k} \binom{k}{i, j} \frac{\frac{\partial^k f}{\partial x^i y^j}(x_0, y_0)}{k!}(x - x_0)^i (y - y_0)^j.$$

Here, the sum is taken over all pairs (i, j) such that $i + j = k$ and k varies from $k = 0$ to $k = n$, and $\binom{k}{i,j}$ is a *multinomial coefficient*, defined by $\binom{k}{i,j} = \frac{k!}{i!j!}$. The multinomial coefficient $\binom{k}{i,j}$ counts the number of times in which the expression $x^i y^j$ appears in the expansion of $(x + y)^k$. This coefficient is needed because e.g. the second degree Talyor polynomial must account for both f_{xy} and f_{yx} (which, for smooth functions, are equal), and all the possible permutations of k-th order patial derivatives appear as terms in the expansion $(x + y)^k$. The expression defined above may then be written more succinctly as

$$P_n(x, y) = \sum_{k=0}^{n} \sum_{i+j=k} \frac{\frac{\partial^k f}{\partial x^i y^j}(x_0, y_0)}{i!j!}(x - x_0)^i (y - y_0)^j.$$

Example 2.2. Let $f(x, y) = e^x \sin(y)$. By the above, the second order Taylor polynomial of $f(x, y)$ at $P = (0, 0)$ is given by

$$P_2(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + \frac{1}{2} f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2} f_{yy}(0, 0)y^2.$$

It's easy to compute $f_x(0, 0) = 0, f_y(0, 0) = 1, f_{xx}(0, 0) = 0, f_{yy}(0, 0) = 0, f_{xy} = 1$, so that

$$P_2(x, y) = y + xy.$$

This is a better approximation to $f(x, y)$ near $(0, 0)$ than the tangent plane approximation!

Now, we state the two dimensional version of Taylor's theorem.

Theorem 2.3 (Taylor)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let $P = (x_0, y_0)$. Then for any $n \geq 1$, there exists a function $h_n(x, y)$ such that*

$$f(x, y) = P_n(x, y) + \sum_{i+j=n} h_n(x, y)(x - x_0)^i (y - y_0)^j$$

and

$$\lim_{(x,y)\to(x_0,y_0)} h_n(x, y) = 0.$$

In one dimension, we have a nice theorem that gives us an upper bound on the error in the Taylor approximation:

Theorem 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. Let $x_0 \in \mathbb{R}$ and $n \geq 1$. Fix $r > 0$, and let M be such that $|f^{(n+1)}(z)| \leq M$ for all $z \in [x_0 - r, x_0 + r]$. Then*

$$|f(x) - P_n(x)| \leq \frac{M|x - x_0|^{n+1}}{(n + 1)!}.$$

The two dimensional analogue is the following:

Theorem 2.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be smooth. Let $P = (x_0, y_0) \in \mathbb{R}^2$ and $n \geq 1$. Fix $r > 0$, and let M be such that for each pair (i, j) with $i + j = n + 1$, $|\frac{\partial^{n+1} f}{\partial x^i \partial y^j}(u, v)| \leq M$ for all (u, v) in the disk $D(P, r)$. Then*

$$|f(x, y) - P_n(x, y)| \leq \frac{M\|(x, y) - (x_0, y_0)\|^{n+1}}{(n + 1)!}.$$

Example 2.6. Let $f(x,y) = e^x \sin(y)$ as before, and suppose we wished to estimate $f(-.1,.1)$ using the second order Taylor polynomial $P_2(x,y) = y + xy$ centered at $P = (0,0)$ computed above. How good is this approximation?

For $i+j=3$, we see that $\frac{\partial^3 f}{\partial x^i y^j}(x,y)$ takes the form $\pm e^x \sin(y)$ or $\pm e^x \cos(y)$. Either way, $|\frac{\partial^3 f}{\partial x^i y^j}(x,y)| \leq e^x$ because $|\cos(y)| \leq 1$ and $|\sin(y)| \leq 1$ always holds. On the disk $D(P,.1)$ the maximum value of e^x is simply $e^{.1}$, which means that $|\frac{\partial^3 f}{\partial x^i y^j}(u,v)| \leq e^{.1}$ is true on $D(P,.1)$. By the error bound formula, this then means that

$$|f(-.1,.1) - P_2(-.1,.1)| \leq \frac{e^{.1}\|(-.1,.1)-(0,0)\|^3}{3!} = \frac{e^{.1}\sqrt{2}}{3000} \approx .0005,$$

which means that $P_2(-.1,.1)$ is at most $.0005$ away from $f(-.1,.1)$. We compute $P_2(-.1,.1) = .1 - .01 = .09$, which means that the true value of $f(-.1,.1)$ lives somewhere in the interval $(.0895,.0905)$ (and in fact, if one uses a calculator they will find $f(-.1,.1) \approx .0903$).