Infinite Series Problems
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Test	Applicable Series	Conclusion	Additional
Divergence	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Always try this first. Inconclusive if $\lim_{n\to\infty} a_n = 0$. Can **not** show convergence!!
Geometric Series	$\sum_{n=M}^{\infty} cr^n$	Converges if $\|r\| < 1$, diverges if $\|r\| \geq 1$	Converges to value $\frac{cr^M}{1-r}$
Direct Comparison	$\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ eventually	If $\sum b_n$ converges, then $\sum a_n$ converges If $\sum a_n$ diverges, then $\sum b_n$ diverges	
Limit Comparison	$\sum a_n$ and $\sum b_n$ with $0 < a_n, b_n$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, $0 < L < \infty$	$\sum a_n$ and $\sum b_n$ both converge or diverge	
Integral	$\sum a_n$ with $a_n = f(n)$ continuous, positive, decreasing eventually for $n \geq M$	$\sum a_n$ and $\int_M^\infty f(x)\,dx$ both converge or diverge	$\|S - S_N\| \leq \int_N^\infty f(x)\,dx$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges if $p > 1$, diverges if $p \leq 1$	
Absolute Convergence	$\sum a_n$	If $\sum \|a_n\|$ converges, $\sum a_n$ converges absolutely	If $\sum a_n$ converges but $\sum \|a_n\|$ diverges, we call this conditional convergence
Ratio	$\sum a_n$ with $a_n \neq 0$ and $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = L$	Converges (absolutely) if $L < 1$, diverges if $L > 1$	Inconclusive if $L = 1$
Root	$\sum a_n$ with $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = L$	Converges (absolutely) if $L < 1$, diverges if $L > 1$	Inconclusive if $L = 1$
Alternating Series	$\sum (-1)^n a_n$ with a_n positive, monotonically decreasing eventually, and $\lim_{n\to\infty} a_n = 0$	$\sum (-1)^n a_n$ converges	$\|S - S_N\| \leq a_{N+1}$

*For series where the starting index of summation does not matter, we simply write $\sum a_n$.

Strategies for testing series

- The first thing to check is if $\lim_{n\to\infty} a_n = 0$, to see if it is even possible for the series to converge.
- Check if the series has a special form. Does it look geometric? Is it a p-series? In these cases, we know exactly how the series behaves. Does it alternate? Try the alternating series test. If the series has negative terms but is not alternating, try seeing if it converges absolutely. Even if the series is alternating, this is sometimes faster.
- If the series involves factorials or n-th powers try the ratio test. The root test isn't great to use when you see factorials – where it really shines is when you see expressions of the form n^n, or exponents with other powers of n. If you can't think of a test to try, the ratio test is rarely a bad idea.
- If a_n is bounded, try a direct/limit comparison test. Analyze the behavior of a_n as n approaches infinity. This will give you other series to compare with either directly or using the limit comparison test. In general, these are the most useful tests.
- If a_n looks simple to integrate, try the integral test. If you can't tell if a_n is something that can be integrated or not, try this last.
- If a_n is a rational function, try using partial fractions to see if the series telescopes to calculate the value of the series.
- If all else fails, check to see if the series is a cleverly disguised telescoping series.

Determine the (conditional) convergence or divergence of the following infinite series, and if possible, compute the value of the sum. Starred problems are challenges.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n - 1 + \cos^2(n^3)}$

2. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln(\ln(n))}$

3.* $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ where $p > 0$ is an arbitrary constant.

4. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n^{2n}}{(1+n^2)^n}$

5. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

6. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1+1/n}}$

7. $\displaystyle\sum_{n=1}^{\infty} \left(1 - \sqrt{1 - \frac{1}{n}}\right)$ *(Hint: multiply and divide by the conjugate expression)*

8.* $\displaystyle\sum_{n=2}^{\infty} \frac{\log_n(n!)}{n^3}$ where $\log_n$ means the base n logarithm.

9. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos(n)}{1 + \ln(n) + n^2 + 2^n + n!}$

10.* $\displaystyle\sum_{n=2}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$

11.* $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{(n+1)^n} \left(\frac{19}{7}\right)^n$

12. $\displaystyle\sum_{n=0}^{\infty} n e^{-n^3}$

13.* $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\sqrt{n^2+n} - n\right)}{n}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 2^{n-3} 3^{2n+1}}{5^{3n}}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 \sin(n + \ln(n))}{n^4 + 4^n}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10} + 11^n}{n^{11} + 10^n}$

17.* $\displaystyle\sum_{n=1}^{\infty} (1 - e^{-n})^n$

18. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

19. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)^3}$

20. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$

22.* $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{e^{n^2}}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + n^2 - \ln(n)}{n!}$

25. $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt[n]{n} - 1)^n$

Solutions

The solutions presented below are what I personally thought was the easiest approach – there are certainly other ways to approach these problems. (If you find easier solutions let me know!)

1. As $n \to \infty$, the term in the denominator that makes the biggest contribution is 2^n. Therefore we expect $\sum_{n=0}^{\infty} \frac{1}{2^n - 1 + \cos^2 n^3}$ and $\sum_{n=0}^{\infty} \frac{1}{2^n}$ to have the same behavior, and the latter is a convergent geometric series. Using the limit comparison test with $a_n = \frac{1}{2^n - 1 + \cos^2 n^3}$ and $b_n = \frac{1}{2^n}$, we have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1 + \cos^2 n^3} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n} + \frac{\cos^2 n^3}{2^n}} = 1$. Therefore by the limit comparison test, $\sum_{n=0}^{\infty} \frac{1}{2^n - 1 + \cos^2 n^3}$ $\boxed{\text{converges}}$.

2. As $\ln n < n$ for all $n \geq 3$, we then have $\ln \ln n < \ln n < n$ for all $n \geq 3$ as well, so that $\frac{1}{n} < \frac{1}{\ln \ln n}$ for $n \geq 3$. Since $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges, by a direct comparison test we see $\sum_{n=3}^{\infty} \frac{1}{\ln \ln n}$ $\boxed{\text{diverges}}$.

3. Here we use the integral test. The function $f(x) = \frac{\ln x}{x^p}$ has derivative $f'(x) = (1 - p\ln x)x^{-p-1}$, which is negative for $x > e^{1/p}$. Let N be a integer larger than $e^{1/p}$. Then $\sum_{n=N}^{\infty} \frac{\ln n}{n^p}$ and $\int_N^{\infty} \frac{\ln x}{x^p}\,dx$ have the same behavior. The latter integral can be done using integration by parts, with $u = \ln x$ and $dv = x^{-p}$. We find $\int_N^{\infty} \frac{\ln x}{x^p}\,dx = \left.\frac{x^{1-p}\ln x}{1-p}\right|_N^{\infty} - \frac{1}{1-p}\int_N^{\infty} x^{-p}\,dx = \left.\frac{x^{1-p}\ln x}{1-p}\right|_N^{\infty} - \left.\frac{x^{1-p}}{(1-p)^2}\right|_N^{\infty} = \left.\frac{x^{1-p}((1-p)\ln x - 1)}{(1-p)^2}\right|_N^{\infty} = \lim_{R\to\infty} \frac{R^{1-p}((1-p)\ln R - 1)}{(1-p)^2} - \frac{N^{1-p}((1-p)\ln N - 1)}{(1-p)^2}$. If $0 < p < 1$, we see the limit is infinite. If $p > 1$, using L'Hopital's rule (or squeeze theorem) shows the limit exists. What happens for $p = 1$? In the case, we care about $\int_N^{\infty} \frac{\ln x}{x}\,dx$, and this is seen to diverge using a u-substitution. Therefore by the integral test, we have the following behavior: $\boxed{\begin{cases} \text{converges} & p > 1 \\ \text{diverges} & 0 < p \leq 1 \end{cases}}$.

4. Note that $\frac{n^{2n}}{(1+n^2)^n} = \left(\frac{n^2}{1+n^2}\right)^n$. Set $L = \lim_{n\to\infty} \left(\frac{n^2}{1+n^2}\right)^n$, then $\ln L = \lim_{n\to\infty} n \ln \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{\ln \frac{n^2}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2\ln n - \ln(n^2+1)}{\frac{1}{n}}$. The second equality shows this is an indeterminate form of type $\frac{0}{0}$, so using L'Hopital's rule on the third equality gives $\lim_{n\to\infty} \frac{2\ln n - \ln(n^2+1)}{\frac{1}{n}} \overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{\frac{2}{n} - \frac{2n}{n^2+1}}{-\frac{1}{n^2}} = \lim_{n\to\infty} -\frac{2n^2}{n(n^2+1)} = 0$. This then gives $L = 1$, so that $\lim_{n\to\infty} (-1)^n \frac{n^{2n}}{(1+n^2)^n}$ doesn't exist (it oscillates between 1 and -1). Therefore, the series $\boxed{\text{diverges}}$ by the divergence test.

5. Here we use the ratio test. With $a_n = \frac{(n!)^2}{(2n)!}$, we have $\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n)!}{(2n+2)!} = \frac{(n+1)^2}{(2n+1)(2n+2)}$. It's then easy to check that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{4} < 1$, so by the ratio test, $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$ $\boxed{\text{converges}}$.

6. As $n \to \infty$, $1 + 1/n \to 1$, so we expect $n^{1+1/n} \approx n$, and therefore that $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ have the same behavior, and the latter we know is the divergent harmonic series. With $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$, we have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{1+1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$. Therefore by the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.

   For the alternating series, one can compute that with $f(x) = frac1x^{1+1/x}$ we have $f'(x) = (\ln(x)-1-x)x^{-1/x-3} < 0$ for $x > 1$, so a_n is eventually decreasing. It's clear that $a_n \to 0$, so by the alternating series test, $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ converges, and so the series $\boxed{\text{converges conditionally}}$.

7. Following the hint, $\left(1 - \sqrt{1 - \frac{1}{n}}\right)\left(1 + \sqrt{1 - \frac{1}{n}}\right) = \frac{1}{n}$, so that $1 - \sqrt{1 - \frac{1}{n}} = \frac{\frac{1}{n}}{1 + \sqrt{1 - \frac{1}{n}}}$. As $1 + \sqrt{1 - \frac{1}{n}} \le 2$, we find $1 - \sqrt{1 - \frac{1}{n}} \ge \frac{\frac{1}{n}}{2} = \frac{1}{2n}$. Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, by a direct comparison we see that $\sum_{n=1}^{\infty} \left(1 - \sqrt{1 - \frac{1}{n}}\right)$ also $\boxed{\text{diverges}}$.

8. Firstly, write $\log_n(n!)$ as $\frac{\ln(n!)}{\ln(n)}$ using the change of base formula, so we can rewrite the series as $\sum_{n=2}^{\infty} \frac{\ln(n!)}{n^3 \ln n}$. Since $n! \le n^n$ for all n, $\sum_{n=2}^{\infty} \frac{\ln(n!)}{n^3 \ln n} \le \sum_{n=2}^{\infty} \frac{\ln(n^n)}{n^3 \ln n} = \sum_{n=2}^{\infty} \frac{n \ln(n)}{n^3 \ln(n)} = \sum_{n=2}^{\infty} \frac{1}{n^2}$. The latter is a convergent p-series, so by direct comparison, $\sum_{n=2}^{\infty} \frac{\ln(n!)}{n^3 \ln n}$ $\boxed{\text{converges}}$.

9. Firstly, recall that $|\cos(n)| \le 1$ for all n. Taking absolute values, $\sum_{n=0}^{\infty} \frac{|\cos(n)|}{1 + \ln(n) + n^2 + 2^n + n!} \le \sum_{n=0}^{\infty} \frac{1}{n!}$. The latter sum is easily seen to converge by the ratio test, so $\sum_{n=0}^{\infty} \frac{\cos(n)}{1 + \ln(n) + n^2 + 2^n + n!}$ $\boxed{\text{converges absolutely}}$.

10. The idea is to analyze the function $\ln(1 + x)$ for $x \approx 0$. Taking a tangent line at $x = 0$, we see that $\ln(1 + x) \approx x$. Since $\frac{1}{n^2} \to 0$ as $n \to \infty$, we see that $\ln\left(1 + \frac{1}{n^2}\right) \approx \frac{1}{n^2}$, so we expect that $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ should have the same behavior. The latter is a convergent p-series, so we expect our original series converges. To prove this, we'll use the limit comparison test. Set $a_n = \ln\left(1 + \frac{1}{n^2}\right)$ and $b_n = \frac{1}{n^2}$. Then $\frac{a_n}{b_n} = \frac{\ln(1 + \frac{1}{n^2})}{\frac{1}{n^2}}$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln(1 + \frac{1}{n^2})}{\frac{1}{n^2}} = \lim_{u\to 0} \frac{\ln(1 + u)}{u}$ using the substitution $u = \frac{1}{n^2}$. We see using L'Hopital's rule that $\lim_{u\to 0} \frac{\ln(1 + u)}{u} = 1$, so by the limit comparison test, we see that $\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n^2}\right)$ $\boxed{\text{converges}}$.

11. Set $a_n = \frac{n!}{(n + 1)^n}\left(\frac{19}{7}\right)^n$. Then $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+2)^{n+1}}\left(\frac{19}{7}\right)^{n+1}}{\frac{n!}{(n+1)^n}\left(\frac{19}{7}\right)^n} = \frac{19}{7} \frac{(n + 1)(n + 1)^n}{(n + 2)^{n+1}} = \frac{19}{7}\left(\frac{n + 1}{n + 2}\right)^n$. Taking limits, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{19}{7}\left(\frac{n + 1}{n + 2}\right)^n$. To evaluate this limit, set $u = n + 2$, so this

becomes $\lim_{u\to\infty} \frac{19}{7}\left(\frac{u-1}{u}\right)^{u-2} = \frac{19}{7}\lim_{u\to\infty}\left(1-\frac{1}{u}\right)^u \cdot \left(1-\frac{1}{u}\right)^{-2} = \frac{19}{7e} < 1$. By the ratio test, $\sum_{n=0}^{\infty} \frac{n!}{(n+1)^n}\left(\frac{19}{7}\right)^n$ $\boxed{\text{converges}}$.

12. Use the root test: set $a_n = \sqrt[n]{\frac{n}{e^{n^3}}} = \frac{n^{1/n}}{e^{n^2}}$. It's an easy limit computation to see that $n^{1/n} \to 1$ as $n\to\infty$, so that $\lim_{n\to\infty} a_n = 0$. Therefore by the root test, $\sum_{n=1}^{\infty}\frac{n}{e^{n^3}}$ $\boxed{\text{converges}}$.

13. Since $\sin(\sqrt{n^2+n}-n)$ is bounded by 1, we'll try replacing the numerator with that. Use a limit comparison test with $a_n = \frac{\sin(\sqrt{n^2+n}-n)}{n}$ and $b_n = \frac{1}{n}$. Then $\frac{a_n}{b_n} = \sin(\sqrt{n^2+n}-n)$, so we need to see how this behaves as $n\to\infty$. We use the trick of multiplying and dividing by the conjugate expression from earlier. Notice that $(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n) = n$, so $\sqrt{n^2+n}-n = \frac{n}{\sqrt{n^2+n}+n}$. This gives $\sin(\sqrt{n^2+n}-n) = \sin\left(\frac{n}{\sqrt{n^2+n}+n}\right)$. One can check with their favorite limit technique that $\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n} = \frac{1}{2}$, so that $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\sin(\sqrt{n^2+n}-n) = \sin\left(\frac{1}{2}\right)$. As $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, by the limit comparison test $\sum_{n=1}^{\infty}\frac{\sin(\sqrt{n^2+n}-n)}{n}$ $\boxed{\text{diverges}}$.

14. We can use some algebra to rewrite the sum as $\frac{3}{8}\sum_{n=1}^{\infty}\frac{(-1)^n 2^n 3^{2n}}{5^{3n}} = \frac{3}{8}\sum_{n=1}^{\infty}\left(\frac{-18}{125}\right)^n = \frac{3}{8}\cdot \frac{-18}{143} = \boxed{\frac{-27}{572}}$ using the formula for the sum of a geometric series.

15. Notice that $|\sin(n+\ln(n))| \le 1$, so we will test for absolute convergence. We have $\sum_{n=1}^{\infty}\frac{n^2|\sin(n+\ln(n))|}{n^4+4^n} \le \sum_{n=1}^{\infty}\frac{n^2}{n^4+4^n} \le \sum_{n=1}^{\infty}\frac{n^2}{n^4} = \sum_{n=1}^{\infty}\frac{1}{n^2}$. The latter sum is a convergent p-series with $p=2$, so by the comparison test, $\sum_{n=1}^{\infty}\frac{n^2\sin(n+\ln(n))}{n^4+4^n}$ $\boxed{\text{converges absolutely}}$.

16. As $n\to\infty$ we notice that $\frac{n^{10}+11^n}{n^{11}+10^n} \approx \frac{11^n}{10^n} = \left(\frac{11}{10}\right)^n$. Since $\sum_{n=1}^{\infty}\left(\frac{11}{10}\right)^n$ is a divergent geometric series, we expect our series diverges as well. Use the limit comparison test with $a_n = \frac{n^{10}+11^n}{n^{11}+10^n}$ and $b_n = \left(\frac{11}{10}\right)^n$, then $\frac{a_n}{b_n} = \frac{10^n n^{10}+110^n}{11^n n^{11}+110^n}$, and dividing through numerator and denominator by 110^n says $\frac{a_n}{b_n} = \frac{\frac{n^{10}}{11^n}+1}{\frac{n^{11}}{10^n}+1}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$, so by the limit comparison test, we see that $\sum_{n=1}^{\infty}\frac{n^{10}+11^n}{n^{11}+10^n}$ $\boxed{\text{diverges}}$.

17. Since $e^n > n$, we have $e^{-n} < \frac{1}{n}$, so that $1-e^{-n} > 1-\frac{1}{n}$. This says $(1-e^{-n})^n > (1-\frac{1}{n})^n$. Taking $n\to\infty$, the right hand side tends to $\frac{1}{e}$, so that $\lim_{n\to\infty}(1-e^{-n})^n \ge \frac{1}{e}$. In particular, since $\lim_{n\to\infty}(1-e^{-n})^n \ne 0$, by the divergence test, we find that $\sum_{n=1}^{\infty}(1-e^{-n})^n$ $\boxed{\text{diverges}}$.

18. Set $a_n = f(n) = \sin\left(\frac{1}{n}\right)$. Then $f'(n) = -\cos\left(\frac{1}{n}\right)\frac{1}{n^2} < 0$ for all $n > 0$, so that a_n is monotonically decreasing. It's clear that $a_n \ge 0$ and $\lim_{n\to\infty} a_n = 0$, so by the alternating series

test, $\sum_{n=1}^{\infty}(-1)^n \sin\left(\frac{1}{n}\right)$ converges. To check for conditional convergence, we look at the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$. Since $\sin(x) \approx x$ when $x \approx 0$, this says $\sin\left(\frac{1}{n}\right) \approx \frac{1}{n}$, so $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \approx \sum_{n=1}^{\infty} \frac{1}{n}$. The latter series is the divergent harmonic series, so we think our series should diverge too. Use the limit comparison test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1$ where we have used the substitution $u = \frac{1}{n}$. Therefore by the limit comparison test, $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ diverges, so our original series $\boxed{\text{converges conditionally}}$.

19. Since $\ln(n) \leq n^a$ eventually for any $a > 0$, with $a = 1/3$ we see that $\ln(n) \leq n^{1/3}$ for all $n \geq N$ for some N, so cubing gives $\ln(n)^3 \leq n$ for $n \geq N$. This then says $\sum_{n=N}^{\infty} \frac{1}{\ln(n)^3} \geq \sum_{n=N}^{\infty} \frac{1}{n}$. The latter series is the divergent harmonic series, so by a direct comparison test, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^3}$ $\boxed{\text{diverges}}$.

20. Use the root test with $a_n = \frac{1}{\ln(n)^n}$. We have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$, so the series $\boxed{\text{converges}}$.

21. Notice that $\ln n > 2$ for $n > e^2$. Then $n^{\ln n} > n^2$ and therefore $\frac{1}{n^{\ln n}} < \frac{1}{n^2}$ for $n \geq 9$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, by a direct comparison this says $\sum_{n=1}^{\infty} \frac{1}{n^{\ln n}}$ $\boxed{\text{converges}}$.

22. Set $a_n = \sqrt{n+1} - \sqrt{n}$. Then using the identity $a - b = \frac{a^2 - b^2}{a + b}$ with $a = \sqrt{n+1}$ and $b = \sqrt{n}$, we see we can write $a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$. Obviously, $\lim_{n\to\infty} a_n = 0$, and $\frac{a_{n+1}}{a_n} = \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n+1}} < 1$, which says a_n is monotonically decreasing. By the alternating series test, we find that $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1} - \sqrt{n})$ converges. We now need to test for conditional convergence. Taking an absolute value, we study the series $\sum_{n=1} \frac{1}{\sqrt{n+1} + \sqrt{n}}$. As $n \to \infty$, $\sqrt{n+1} \approx \sqrt{n}$, so we expect the series looks like $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$, which we see is a divergent p-series with $p = 1/2$. Setting $b_n = \frac{1}{2\sqrt{n}}$, we see $\frac{a_n}{b_n} = \frac{2\sqrt{n}}{\sqrt{n} + \sqrt{n+1}}$, and it's easy to check that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Therefore by the limit comparison test, $\sum_{n=1} \frac{1}{\sqrt{n+1} + \sqrt{n}}$ diverges so that $\sum_{n=1}^{\infty}(-1)^n(\sqrt{n+1} - \sqrt{n})$ $\boxed{\text{converges conditionally}}$.

23. We use the ratio test. Set $a_n = \frac{n!}{e^{n^2}}$. Then $a_{n+1} = \frac{(n+1)!}{e^{n^2+2n+1}}$, so $\frac{a_{n+1}}{a_n} = \frac{n+1}{e^{2n+1}}$. It's then clear using L'Hopital's rule that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, so by the ratio test, the series $\boxed{\text{converges}}$.

24. As $n \to \infty$, $2^n + n^2 - \ln(n) \approx 2^n$, so $\frac{2^n + n^2 - \ln(n)}{n!} \approx \frac{2^n}{n!}$. Since $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is easily seen to converge with the ratio test, we expect our original series converges as well. Using the limit comparison test with $a_n = \frac{2^n + n^2 - \ln(n)}{n!}$ and $b_n = \frac{2^n}{n!}$, we have $\frac{a_n}{b_n} = \frac{2^n + n^2 - \ln(n)}{2^n} =$

$1 + \dfrac{n^2}{2^n} - \dfrac{\ln(n)}{2^n}$, so as $n \to \infty$, we see that $\dfrac{a_n}{b_n} \to 1$. By the limit comparison test, this says $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + n^2 - \ln(n)}{n!}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ have the same behavior, so our series $\boxed{\text{converges}}$.

25. We use the root test since there are n-th powers floating around instead of the alternating series test. Set $a_n = (-1)^n(\sqrt[n]{n} - 1)^n$, so that $\sqrt[n]{|a_n|} = |a_n|^{1/n} = \sqrt[n]{n} - 1 = n^{1/n} - 1$. As $n \to \infty$, it's a standard exercise in taking logarithms to see that $n^{1/n} \to 1$, so $n^{1/n} - 1 \to 0$. By the root test, the series $\boxed{\text{converges absolutely}}$.