Integration Practice Problems

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Starred problems are challenges.

1. Compute $\int \sqrt{x} \ln(x)\, dx$

2. Compute $\int \frac{1}{\sqrt{x^2 - 4x + 7}}\, dx$

3. Compute $\int \cos^{-1}(x)\, dx$

4. Compute $\int \cos^3(x) \sin^8(x)\, dx$

5. Compute $\int \frac{2x^3 + 2x^2 - 2x + 1}{x^2(x-1)^2}\, dx$

6.* Compute $\int \frac{1}{1 + \sqrt{x+1}}\, dx$

7. Compute $\int x^2 \ln(x^2 + 1)\, dx$

8.* Compute $\int \frac{\sqrt{\tan x}}{\sin 2x}\, dx$

9. Compute $\int \frac{1}{e^x \sqrt{1 - e^{-2x}}}\, dx$

10. Compute $\int \sin(\ln(x))\, dx$

11. Compute $\int \frac{1}{1 + e^x}\, dx$

12. Compute $\int \sec^3(x)\, dx$

13. Compute $\int \frac{1}{(x + x^{-1})^2}\, dx$

14*. Compute $\int \frac{1}{1 + \sin(x)}\, dx$

15.* Compute $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$

16. Compute $\int \sin^4(x) \cos^2(x)\, dx$

17. Compute $\int \ln(x^2 + 2x + 2)\, dx$

18. Compute $\int \frac{x^4}{\sqrt{1 - x^2}}\, dx$

19. Compute $\int \frac{x}{\sqrt{2 - x^4}}\, dx$

20.* Compute $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$

21. Compute $\int \frac{3x^3+6x^2+7}{x^2(x^2+7)}\,dx$

22. Compute $\int \sin^{-1}(\sqrt{x})\,dx$

23. Compute $\int \frac{\ln(x)}{(x+1)^2}\,dx$

24.* Compute $\int \frac{1}{x^4+4}\,dx$ *(Hint: complete the square to factor the denominator.)*

25.* Compute $\int \frac{1-x}{(x^2+x+1)^2}\,dx$

Strategies for integration

- Substitute as much as possible to try and simplify the integrand. If you see nasty looking expressions that you do not want to deal with, it's not a bad idea to "substitute them away".
- There is not always an "obvious" choice of substitution – sometimes you may want to use the entire integrand as a new variable, or just parts of it. If you see multiple difficult expressions, try to write them both in terms of a new expression. Sometimes it is useful to remove radicals by using substitution, or by turning quantities into expressions that you can use trigonometric substitutions on.
- Sometimes algebras tricks are useful. Common ones include adding and subtracting a quantity, and multiplying and dividing by a quantity. Other times, algebraic manipulations are necessary to compute an anti-derivative, like completing a square, or using polynomial long division.
- For integrals involving trig functions, try to use trigonometric identities (or force them to appear).
- When in doubt, try integration by parts. As a general rule of thumb, pick u to be an expression easy to differentiate, and dv to be the most complicated looking term in the integrand that you can easily find an anti-derivative of. If there is no obvious product to use integration by parts on, either make one appear (by multiplying and dividing by some quantity) or trying $dv = 1$.
- Rational functions can be integrated algorithmically using a partial fraction decomposition. If all else fails, try turning the integrand into a rational function via some substitution.

Solutions

As the techniques used for integration are very flexible, there are many different approaches to computing these integrals. Below are the approaches that I though were most natural – you certainly may find a different method to be easier. Many of the answers can be written in various forms (e.g. using inverse hyperbolic trig functions). If you arrive at an answer that looks different and it's not obviously equivalent to something listed here, it's best to check with computational software (like WolframAlpha) that your answer is a valid anti-derivative.

1. Set $t = \sqrt{x}$ so that $dx = 2t\,dt$. The integral becomes $\int 2t^2 \ln(t^2)\,dt = \int 4t^2 \ln(t)\,dt$. Now use integration by parts with $u = \ln(t)$ and $dv = 4t^2$ to get $\int 4t^2 \ln(t)\,dt = \frac{4}{3}t^3 \ln(t) - \int \frac{4}{3}t^2\,dt = \frac{4}{3}t^3 \ln(t) - \frac{4}{9}t^3 + C = \boxed{\frac{2}{3}x^{3/2}\ln(x) - \frac{4}{9}x^{3/2} + C}$.

2. First complete the square to get $\int \frac{1}{\sqrt{(x-2)^2+3}}\,dx$. Set $x - 2 = \sqrt{3}\tan\theta$, so that $dx = \sqrt{3}\sec^2\theta\,d\theta$. Plugging in gives $\int \frac{1}{\sqrt{3}\sec\theta}\sqrt{3}\sec^2\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$. From $x-2 = \sqrt{3}\tan\theta$, this says $\tan\theta = \frac{x-2}{\sqrt{3}}$, so drawing the appropriate right triangle says $\sec\theta = \sqrt{\frac{x^2-4x+7}{3}}$. Plugging in gives $\int \frac{1}{\sqrt{x^2-4x+7}}\,dx = \boxed{\ln\left|\frac{x-2}{\sqrt{3}} + \sqrt{\frac{x^2-4x+7}{3}}\right| + C}$.

3. Integrate by parts with $u = \cos^{-1}(x)$ and $dv = 1$. This gives $\int \cos^{-1}(x)\,dx = x\cos^{-1}(x)\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$. Set $u = 1 - x^2$ to turn $\int \frac{x}{\sqrt{1-x^2}}\,dx$ into $-\frac{1}{2}\int u^{-1/2}\,du = -\sqrt{u} + C = -\sqrt{1-x^2} + C$. This gives $\int \cos^{-1}(x)\,dx = \boxed{x\cos^{-1}(x) - \sqrt{1-x^2} + C}$.

4. Write $\cos^3(x) = \cos(x)\cos^2(x) = \cos(x)(1-\sin^2(x))$. The integral becomes $\int \cos(x)(1-\sin^2(x))\sin^8(x)\,dx$. Set $u = \sin(x)$ so $du = \cos(x)\,dx$. After substitution this equals $\int (1-u^2)u^8\,du = \int u^8 - u^{10}\,du = \frac{1}{9}u^9 - \frac{1}{11}u^{11} + C = \boxed{\frac{1}{9}\sin^9(x) - \frac{1}{11}\sin^{11}(x) + C}$.

5. Do a partial fraction decomposition: $\frac{2x^3+2x^2-2x+1}{x^2(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$. Multiply through by the common denominator to get $2x^3 + 2x^2 - 2x + 1 = Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2$. Plug in $x = 0$ and $x = 1$ to get $B = 1$ and $D = 3$. Expand out the right hand side to get $(A+C)x^3 + (4-2A-C)x^2 + (A-2)x + 1$. Comparing coefficients we have $A + C = 2$ and $A - 2 = 2$. This says $A = 0$ and $C = 2$, so our partial fraction expansion is $\frac{2x^3+2x^2-2x+1}{x^2(x-1)^2} = \frac{1}{x^2} + \frac{2}{x-1} + \frac{3}{(x-1)^2}$. Integrating gives $\int \frac{2x^3+2x^2-2x+1}{x^2(x-1)^2}\,dx = \int \frac{1}{x^2} + \frac{2}{x-1} + \frac{3}{(x-1)^2}\,dx = \boxed{-\frac{1}{x} + 2\ln|x-1| - \frac{3}{x-1} + C}$.

6. Multiply and divide by the conjugate expression $1 - \sqrt{x+1}$ to get $\int \frac{1}{1+\sqrt{x+1}}\,dx = \int \frac{1-\sqrt{x+1}}{-x}\,dx = \int \frac{\sqrt{x+1}-1}{x}\,dx = \int \frac{\sqrt{x+1}}{x}\,dx - \ln|x|$. Set $u = \sqrt{x+1}$. Then as usual, $dx = 2u\,du$ so the integral becomes $\int \frac{2u^2}{u^2-1}\,du$. Add and subtract 2 in the numerator to write this as $\int 2 + \frac{2}{u^2-1}\,du$. Using partial fractions, we can write $\frac{2}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$.

Solving for A and B gives $A = 1$ and $B = -1$ so $\frac{2u^2}{u^2-1} = 2 + \frac{1}{u-1} - \frac{1}{u+1}$. Therefore, $\int \frac{2u^2}{u^2-1}\,du = \int 2 + \frac{1}{u-1} - \frac{1}{u+1}\,du = 2u + \ln|u-1| - \ln|u+1| = 2\sqrt{x+1} + \ln|\sqrt{x+1}-1| - \ln|\sqrt{x+1}+1|$. Combining with the previous term, $\int \frac{1}{1+\sqrt{x+1}}\,dx = \boxed{2\sqrt{x+1} + \ln|\sqrt{x+1}-1| - \ln|\sqrt{x+1}+1| - \ln|x| + C}$.

7. First integrate by parts with $dv = x^2$ and $u = \ln(x^2+1)$ to get $\int x^2\ln(x^2+1)\,dx = \frac{1}{3}x^3\ln(x^2+1) - \frac{2}{3}\int \frac{x^4}{x^2+1}\,dx$. To compute $\int \frac{x^4}{x^2+1}\,dx$, perform long division: $\int \frac{x^4}{x^2+1}\,dx = \int \left(x^2 - 1 + \frac{1}{x^2+1}\right)dx$ (If you don't remember how long division works, you can do the integral via trig substitution). Therefore, $\int \frac{x^4}{x^2+1}\,dx = \frac{1}{3}x^3 - x + \tan^{-1}(x)$. Plugging this in, $\int x^2\ln(x^2+1)\,dx = \boxed{\frac{1}{3}x^3\ln(x^2+1) - \frac{2}{9}x^3 + \frac{2}{3}x - \frac{2}{3}\tan^{-1}(x) + C}$.

8. Use the double angle formula and multiply and divide by $\cos(x)$ to get $\int \frac{\sqrt{\tan(x)}}{\sin(2x)}\,dx = \frac{1}{2}\int \frac{\sqrt{\tan(x)}}{\sin(x)\cos(x)} \cdot \frac{\cos(x)}{\cos(x)}\,dx = \frac{1}{2}\int \sqrt{\tan(x)}\cot(x)\sec^2(x)\,dx$. Set $u = \tan(x)$, so $du = \sec^2(x)\,dx$. The integral becomes $\frac{1}{2}\int \sqrt{u}\frac{1}{u}\,du = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + C = \boxed{\sqrt{\tan(x)} + C}$.

9. Notice that $\int \frac{1}{e^x\sqrt{1-e^{-2x}}}\,dx = \int \frac{1}{e^x\sqrt{\frac{e^{2x}-1}{e^{2x}}}}\,dx = \int \frac{1}{\sqrt{e^{2x}-1}}\,dx$. Set $u = e^x$, then $du = e^x\,dx$ so that $dx = \frac{1}{u}\,du$. Then $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx = \int \frac{1}{u\sqrt{u^2-1}}\,du = \sec^{-1}(u) + C = \boxed{\sec^{-1}(e^x) + C}$.

10. Set $t = \ln(x)$, so that $dt = \frac{1}{x}\,dx = \frac{1}{e^t}\,dx$, giving $dx = e^t\,dt$. The integral then becomes $\int e^t\sin(t)\,dt$. Integrate by parts with $u = e^t$ and $dv = \sin(t)$ to get $\int e^t\sin(t)\,dt = -e^t\cos(t) + \int e^t\cos(t)\,dt$. Integrate by parts again with $u = e^t$ and $dv = \cos(t)$ to get $\int e^t\cos(t)\,dt = e^t\sin(t) - \int e^t\sin(t)\,dt$. Substituting this in we get $\int e^t\sin(t)\,dt = -e^t\cos(t) + e^t\sin(t) - \int e^t\sin(t)\,dt$. This gives $2\int e^t\sin(t)\,dt = -e^t\cos(t) + e^t\sin(t)$, so solving for the integral gives us $\int e^t\sin(t)\,dt = -\frac{e^t\cos(t)}{2} + \frac{e^t\sin(t)}{2} + C$. Substituting back in for t gives us $\boxed{-\frac{x\cos(\ln x)}{2} + \frac{x\sin(\ln x)}{2} + C}$.

11. Set $u = 1 + e^x$ so that $du = e^x\,dx = (u-1)\,dx$. This says $dx = \frac{1}{u-1}\,du$. The integral then becomes $\int \frac{1}{u(u-1)}\,du$. Using partial fractions, write $\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$. Solving for A and B gives $A = -1$ and $B = 1$, so $\int \frac{1}{u(u-1)}\,du = \int -\frac{1}{u} + \frac{1}{u-1}\,du = \int -\frac{1}{u}\,du + \int \frac{1}{u-1}\,du = -\ln|u| + \ln|u-1| + C = \boxed{x - \ln|1+e^x| + C}$.

12. Write this as $\int \sec(x)\sec^2(x)\,dx$. Use integration by parts with $u = \sec(x)$ and $dv = \sec^2(x)$ to get $\int \sec^3(x)\,dx = \sec(x)\tan(x) - \int \tan^2(x)\sec(x)\,dx$. Write $\tan^2(x) = \sec^2(x) - 1$,

so this gives $\int \sec^3(x)\,dx = \sec(x)\tan(x) - \int (\sec^2(x) - 1)\sec(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx - \int \sec^3(x)\,dx$. As $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)|$, we see $\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| - \int \sec^3(x)\,dx$. This then gives $2\int \sec^3(x)\,dx = \sec(x)\tan(x) + \ln|\sec(x) + \tan(x)| + C$, so that $\int \sec^3(x)\,dx = \boxed{\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + C}$.

13. We have $\frac{1}{(x + x^{-1})^2} = \frac{1}{(x + \frac{1}{x})^2} = \frac{1}{(\frac{x^2+1}{x})^2} = \frac{x^2}{(x^2 + 1)^2}$, so $\int \frac{1}{(x + x^{-1})^2}\,dx = \int \frac{x^2}{(x^2 + 1)^2}\,dx$. Integrate by parts with $u = x$ and $dv = \frac{x}{(x^2 + 1)^2}$ to get $\int \frac{x^2}{(x^2 + 1)^2}\,dx = -\frac{x}{2(x^2 + 1)} + \int \frac{1}{2(x^2 + 1)}\,dx = \boxed{\frac{1}{2}\tan^{-1}(x) - \frac{x}{2(x^2 + 1)} + C}$.

14. Multiply and divide by $\frac{1 - \sin(x)}{1 - \sin(x)}$ to get $\int \frac{1}{1 + \sin(x)}\,dx = \int \frac{1 - \sin(x)}{1 - \sin^2(x)}\,dx = \int \frac{1 - \sin(x)}{\cos^2(x)}\,dx$. Split this up as $\int \sec^2(x)\,dx - \int \frac{\sin(x)}{\cos^2(x)}\,dx$. The first integral is just $\tan(x)$, and the second integral can be done with $u = \cos(x)$ and $du = -\sin(x)\,dx$ to turn into $\int \frac{-du}{u^2} = \frac{1}{u} = \sec x$. Combining we get $\int \frac{1}{1 + \sin(x)}\,dx = \boxed{\tan(x) - \sec(x) + C}$.

15. Multiply through by e^x to write the integral as $\int \frac{e^{2x} - 1}{e^{2x} + 1}\,dx$. We want to do a u-sub, so multiply and divide by e^x to make this possible: $\int \frac{e^{2x} - 1}{e^{2x} + 1}\,dx = \int \frac{e^{2x} - 1}{e^{2x} + 1} \cdot \frac{e^x}{e^x}\,dx = \int \frac{u^2 - 1}{u(u^2 + 1)}\,du$ with $u = e^x$. At this point, this becomes a basic partial fractions problem: $\frac{u^2 - 1}{u(u^2 + 1)} = \frac{-1}{u} + \frac{2u}{u^2 + 1}$, so $\int \frac{u^2 - 1}{u(u^2 + 1)}\,du = \int \frac{-1}{u} + \frac{2u}{u^2 + 1}\,du = -\ln|u| + \frac{1}{2}\ln|u^2 + 1| + C = \boxed{-x + \frac{1}{2}\ln|e^{2x} + 1| + C}$.

16. Using the power reduction formulas, $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ and $\cos^2 x = \frac{1 + \cos(2x)}{2}$, the integral turns into $\int \left(\frac{1 - \cos(2x)}{2}\right)^2 \left(\frac{1 + \cos(2x)}{2}\right) dx = \int \left(\frac{1 - 2\cos(2x) + \cos(2x)^2}{4}\right)\left(\frac{1 + \cos(2x)}{2}\right) dx$. After multiplying everything out, this becomes $\frac{1}{8}\int (1 - \cos(2x) - \cos^2(2x) + \cos^3(2x))\,dx$. Splitting this up, this becomes $\frac{1}{8}\int dx - \frac{1}{8}\int \cos(2x)\,dx - \frac{1}{8}\int \cos^2(2x)\,dx + \frac{1}{8}\int \cos^3(2x)\,dx$. The first two integrals are easy. For the third integral, use the power reduction formula again. For the last integral, $\frac{1}{8}\int \cos^3(2x)\,dx = \frac{1}{8}\int \cos(2x)(1 - \sin^2(2x))\,dx = \frac{1}{8}\int \cos(2x)\,dx - \frac{1}{8}\int \sin^2(2x)\cos(2x)\,dx$. The latter integral can be done with the substitution $u = \sin(2x)$. Putting everything together, $\frac{1}{8}\int dx - \frac{1}{8}\int \cos(2x)\,dx - \frac{1}{8}\int \cos^2(2x)\,dx + \frac{1}{8}\int \cos^3(2x)\,dx = \frac{x}{8} - \frac{1}{16}\sin(2x) - (\frac{x}{16} + \frac{1}{64}\sin(4x)) + (\frac{1}{16}\sin(2x) - \frac{1}{48}\sin^3(2x)) + C$, so that $\int \sin^4(x)\cos^2(x)\,dx = \boxed{\frac{x}{16} - \frac{1}{64}\sin(4x) - \frac{1}{48}\sin^3(2x) + C}$.

17. Complete the square to write $\int \ln(x^2 + 2x + 2)\,dx = \int \ln((x + 1)^2 + 1)\,dx$. Set $t = x + 1$, so the integral becomes $\int \ln(t^2 + 1)\,dt$. Integrate by parts with $dv = 1$ and $u = \ln(t^2 + 1)$ to get

$\int \ln(t^2+1)\,dt = t\ln(t^2+1) - \int \frac{2t^2}{t^2+1}\,dt$. To evaluate the latter integral, add and subtract to get $\int \frac{2t^2}{t^2+1}\,dt = \int \frac{2t^2+2}{t^2+1} - \frac{2}{t^2+1}\,dt = 2\int dt - \int \frac{2}{t^2+1}\,dt = 2t - 2\tan^{-1}(t) + C$. Putting everything together gives $\int \ln(t^2+1)\,dt = t\ln(t^2+1) - 2t + 2\tan^{-1}(t) + C = (x+1)\ln(x^2+2x+2) - 2(x+1) + 2\tan^{-1}(x+1) + C = \boxed{(x+1)\ln(x^2+2x+2) - 2x + 2\tan^{-1}(x+1) + C}$.

18. Set $x = \sin(\theta)$ so $dx = \cos(\theta)\,d\theta$. The integral becomes $\int \sin^4(\theta)\,d\theta$. Now we use the power reduction formula: $\sin^4(\theta) = (\sin^2(\theta))^2 = (\frac{1}{2}(1-\cos(2\theta)))^2 = \frac{1}{4}(1-\cos(2\theta))^2 = \frac{1}{4}(1 - 2\cos(2\theta) + \cos^2(2\theta))$. Using the power reduction formula again, $\frac{1}{4}(1 - 2\cos(2\theta) + \cos^2(2\theta)) = \frac{1}{4}(1 - 2\cos(2\theta) + \frac{1}{2}(1+\cos(4\theta))) = \frac{3}{8} - \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta)$. This gives $\int \sin^4(\theta)\,d\theta = \int \frac{3}{8}\,d\theta - \frac{1}{2}\int \cos(2\theta)\,d\theta + \frac{1}{8}\int \cos(4\theta)\,d\theta = \frac{3}{8}\theta - \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)$. From the double-angle formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$, we see $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 4\sin(\theta)\cos(\theta)(\cos^2(\theta) - \sin^2(\theta))$. Recalling that $\sin(\theta) = x$, drawing the appropriate triangle gives $\theta = \sin^{-1}(x)$ and $\cos(\theta) = \sqrt{1-x^2}$. Plugging this all in says $\sin(4\theta) = 4x\sqrt{1-x^2}(1-2x^2)$. We do this for the other terms to get $\int \frac{x^4}{\sqrt{1-x^2}}\,dx = \frac{3}{8}\sin^{-1}(x) - \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{8}x\sqrt{1-x^2}(1-2x^2) + C = \boxed{\frac{3}{8}\sin^{-1}(x) - \frac{1}{8}x\sqrt{1-x^2}(2x^2+3) + C}$.

19. Set $u = x^2$, so $du = 2x\,dx$. The integral becomes $\frac{1}{2}\int \frac{du}{\sqrt{2-u^2}} = \frac{1}{2}\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + C = \boxed{\frac{1}{2}\sin^{-1}\left(\frac{x^2}{\sqrt{2}}\right) + C}$.

20. Set $t = x^{1/6}$, so that $dt = \frac{1}{6}x^{-5/6}\,dx$ says $6t^5\,dt = dx$. Then $t^3 = \sqrt{x}$ and $t^2 = \sqrt[3]{x}$, so the $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int \frac{6t^5}{t^3+t^2}\,dt$. Diving through by t^2, this becomes $\int \frac{6t^3}{t+1}\,dt$. Using long division, this becomes $\int 6t^2 - 6t + 6 - \frac{6}{t+1}\,dt = 2t^3 - 3t^2 + 6t - \ln|t+1| + C$. Swapping back to the variable x, we find $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \boxed{2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - \ln|\sqrt[6]{x}+1| + C}$.

21. Using partial fractions, write $\frac{3x^3+6x^2+7}{x^2(x^2+7)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+7}$. Clearing denominators, $3x^3+6x^2+7 = Ax(x^2+7) + B(x^2+7) + (Cx+D)x^2$, and expanding out and collecting the terms gives $3x^3+6x^2+7 = (A+C)x^3 + (B+D)x^2 + 7Ax + 7B$. By comparing coefficients, $A = 0$, $B = 1$, $C = 3$ and $D = 5$. This gives $\frac{3x^3+6x^2+7}{x^2(x^2+7)} = \frac{1}{x^2} + \frac{3x+5}{x^2+7}$, so $\int \frac{3x^3+6x^2+7}{x^2(x^2+7)}\,dx = \int \frac{1}{x^2} + \frac{3x+5}{x^2+7}\,dx$. The first term is easy and integrates to $-\frac{1}{x}$. To compute $\int \frac{3x+5}{x^2+7}\,dx$, split this up as $\int \frac{3x}{x^2+7}\,dx + \int \frac{5}{x^2+7}\,dx$. Using $u = x^2+7$ the first integral becomes $\frac{3}{2}\int \frac{1}{u}\,du = \frac{3}{2}\ln|u| = \frac{3}{2}\ln|x^2+7|$. The second we recognize as an arctangent integral and evaluates to $\frac{5}{\sqrt{7}}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right)$, so putting everything together we get $\int \frac{3x^3+6x^2+7}{x^2(x^2+7)}\,dx = \boxed{-\frac{1}{x} + \frac{3}{2}\ln(x^2+7) + \frac{5}{\sqrt{7}}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C}$.

22. Set $t = \sqrt{x}$, so that $2t\,dt = dx$ turning the integral into $\int 2t\sin^{-1}(t)\,dt$. Integrate by parts

with $u = \sin^{-1}(t)$ and $dv = 2t$, to get $\int 2t\sin^{-1}(t)\,dt = t^2\sin^{-1}(t) - \int \frac{t^2}{\sqrt{1-t^2}}\,dt$. Write $t = \sin(\theta)$, so $dt = \cos(\theta)\,d\theta$. Then $\int \frac{t^2}{\sqrt{1-t^2}}\,dt = \int \sin^2(\theta)\,d\theta$. Use the power reduction formula to write $\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$, so $\int \sin^2(\theta)\,d\theta = \int \frac{1}{2}\,d\theta - \int \frac{1}{2}\cos(2\theta)\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) = \frac{1}{2}\theta - \frac{1}{2}\sin(\theta)\cos(\theta)$. Since $\sin(\theta) = t$, drawing the appropriate right triangle says $\cos(\theta) = \sqrt{1-t^2}$, so that in terms of t the integral becomes $\frac{1}{2}\sin^{-1}(t) - \frac{t}{2}\sqrt{1-t^2}$. Combining with the previous term, $\int 2t\sin^{-1}(t)\,dt = t^2\sin^{-1}(t) - \frac{1}{2}\sin^{-1}(t) + \frac{t}{2}\sqrt{1-t^2} + C$. Substituting back in for x, we find $\int \sin^{-1}(\sqrt{x})\,dx = \boxed{x\sin^{-1}(\sqrt{x}) - \frac{1}{2}\sin^{-1}(\sqrt{x}) + \frac{\sqrt{x}}{2}\sqrt{1-x} + C}$.

23. First, integrate by parts with $u = \ln(x)$ and $dv = \frac{1}{(x+1)^2}$. Then $du = \frac{1}{x}$ and $v = \frac{-1}{x+1}$. We then find $\int \frac{\ln(x)}{(x+1)^2}\,dx = -\frac{\ln(x)}{x+1} + \int \frac{1}{x(x+1)}\,dx$. For the second integral, use partial fractions to write $\int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x} - \frac{1}{x+1}\,dx = \ln|x| - \ln|x+1|$. Putting this together, we find $\int \frac{\ln(x)}{(x+1)^2}\,dx = \boxed{-\frac{\ln(x)}{x+1} + \ln\left|\frac{x}{x+1}\right| + C}$.

24. $x^4+4 = (x^2)^2 + (2)^2$, so completing the square says $x^4+4 = (x^2+2)^2 - 4x^2 = (x^2+2)^2 - (2x)^2 = (x^2+2x+2)(x^2-2x+2)$. Write $\frac{1}{x^4+4} = \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{x^2-2x+2}$. Clearing denominators says $1 = (Ax+B)(x^2-2x+2) + (Cx+D)(x^2+2x+2)$, and expanding out gives $(A+C)x^3 + (-2A+B+2C+D)x^2 + (2A-2B+2C+2D)x + (2B+2D) = 1$. Comparing coefficients, we get $A+C = 0$, $-2A+B+2C+D = 0$, $2A-2B+2C+2D = 0$, and $2B+2D = 1$. The first equation says $A = -C$, so plugging to the third says $2D-2B = 0$, i.e. $B = D$. Then the last equation says $4B = 1$, so $B = D = 1/4$. The second equation says $4C + 1/2 = 0$, so $C = -1/8$ and $A = 1/8$. This gives $\frac{1}{x^4+4} = \frac{\frac{1}{8}x+\frac{1}{4}}{x^2+2x+2} + \frac{-\frac{1}{8}x+\frac{1}{4}}{x^2-2x+2}$ as a partial fraction decomposition. Integrating gives $\int \frac{1}{x^4+4}\,dx = \int \frac{\frac{1}{8}x+\frac{1}{4}}{x^2+2x+2} + \frac{-\frac{1}{8}x+\frac{1}{4}}{x^2-2x+2}\,dx = \int \frac{\frac{1}{8}x+\frac{1}{4}}{x^2+2x+2}\,dx + \int \frac{-\frac{1}{8}x+\frac{1}{4}}{x^2-2x+2}\,dx$. For the first integral, complete the square and pull out the constant to write this as $\frac{1}{8}\int \frac{x+2}{(x+1)^2+1}\,dx$. Set $u = x+1$, then the integral becomes $\frac{1}{8}\int \frac{u+1}{u^2+1}\,du = \frac{1}{8}\int \frac{u}{u^2+1}\,du + \frac{1}{8}\int \frac{1}{u^2+1}\,du = \frac{1}{16}\ln|u^2+1| + \frac{1}{8}\tan^{-1}(u) = \frac{1}{16}\ln|x^2+2x+2| + \frac{1}{8}\tan^{-1}(x+1)$. Similarly, the second integral can be written as $\frac{1}{8}\int \frac{-x+2}{(x-1)^2+1}\,dx$ and setting $u = x-1$, this becomes $\frac{1}{8}\int \frac{1-u}{u^2+1}\,du = \frac{1}{8}\tan^{-1}(u) - \frac{1}{16}\ln|u^2+1| = \frac{1}{8}\tan^{-1}(x-1) - \frac{1}{16}\ln|x^2-2x+2|$. Adding these together, we see

$$\int \frac{1}{x^4+4}\,dx = \boxed{\frac{1}{16}\ln|x^2+2x+2| - \frac{1}{16}\ln|x^2-2x+2| + \frac{1}{8}\tan^{-1}(x+1) - \frac{1}{8}\tan^{-1}(x-1) + C}.$$

25. Add and subtract $\int \frac{3x}{(x^2+x+1)^2}\,dx$ to get $\int \frac{1-x}{(x^2+x+1)^2}\,dx = \int \frac{2x+1}{(x^2+x+1)^2}\,dx - \int \frac{3x}{(x^2+x+1)^2}\,dx$. With $u = x^2+x+1$ and $du = (2x+1)\,dx$ the first integral becomes $\int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{x^2+x+1}$. For the second integral, complete the square and set $u = x + 1/2$ to turn this into $\int \frac{3(u-1/2)}{(u^2+3/4)^2}\,du = \int \frac{3u}{(u^2+3/4)^2}\,du - \frac{3}{2}\int \frac{1}{(u^2+3/4)^2}\,du$. For the first

of these integrals, set $v = u^2 + 3/4$ so $dv = 2u\, du$ making the first integral turn into $\frac{3}{2}\int \frac{dv}{v^2} = -\frac{3}{2v} = -\frac{3}{2(x^2+x+1)}$. This leaves one integral left, $\frac{3}{2}\int \frac{1}{(u^2+3/4)^2}\, du$. Set $u = \frac{\sqrt{3}}{2}\tan(\theta)$ so that $du = \frac{\sqrt{3}}{2}\sec^2(\theta)\, d\theta$. This turns the integral into $\frac{4\sqrt{3}}{3}\int \cos^2(\theta)\, d\theta$. Using the power reduction formula, this turns into $\frac{4\sqrt{3}}{3}\int \frac{1}{2}(1+\cos(2\theta))\, d\theta = \frac{2\sqrt{3}}{3}\int 1+\cos(2\theta)\, d\theta = \frac{2\sqrt{3}}{3}\theta + \frac{\sqrt{3}}{3}\sin(2\theta) = \frac{2\sqrt{3}}{3}\theta + \frac{2\sqrt{3}}{3}\sin(\theta)\cos(\theta)$. Since $\tan(\theta) = \frac{2u}{\sqrt{3}}$, drawing the appropriate right triangle says $\sin(\theta) = \frac{2u}{\sqrt{4u^2+3}}$ and $\cos(\theta) = \frac{\sqrt{3}}{\sqrt{4u^2+3}}$. Substituting the relevant values gives $\frac{3}{2}\int \frac{1}{(u^2+3/4)^2}\, du = \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + \frac{4u}{4u^2+3} = \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{(2x+1)}{\sqrt{3}}\right) + \frac{4x+2}{4(x^2+x+1)}$. Combining everything together gives

$$\int \frac{1-x}{(x^2+x+1)^2}\, dx = \boxed{\frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{(2x+1)}{\sqrt{3}}\right) + \frac{x+1}{x^2+x+1} + C}.$$